

Quasi-Hopf \ast -Algebras

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Abstract

We introduce quasi-Hopf \ast -algebras i.e. quasi-Hopf algebras equipped with a conjugation (star) operation. The definition of quasi-Hopf \ast -algebras proposed ensures that the class of quasi-Hopf \ast -algebras is closed under twisting and additionally, that any Hopf \ast -algebra becomes a quasi-Hopf \ast -algebra via twisting. The basic properties of these algebras are developed. The relationship between the antipode and star structure is investigated. Quasi-triangular quasi-Hopf \ast -algebras are introduced and studied.

1 Introduction

Many of the ideas and constructions from the theory of Hopf algebras have analogues in the quasi-Hopf algebra setting. Examples include the quantum double construction [3, 16, 21, 24], the Tannaka-Krein theorem [22], the existence of integrals [4, 15, 25], the construction of link invariants [1, 18] and extension to the superalgebra case [14, 29], amongst others.

In the Hopf algebra setting, Hopf algebras that admit a conjugation or star operation are well known [19, 26, 28] and are called \ast -Hopf algebras or Hopf \ast -algebras. The introduction of a \ast -structure is not part of the algebraic formalism of Hopf algebras but becomes necessary for physical applications, such as in quantum mechanics where unitarity is a requirement. The notion of a \ast -structure has been extended to the weak Hopf algebra case [2] and also to the braided setting [23], but the quasi-Hopf algebra case appears to have been neglected in the literature. Quasi-Hopf algebras have applications in conformal field theory [5, 6] and in the theory of integrable models (via elliptic quantum groups [8, 9, 10, 11, 17, 29]). It is expected that \ast -structures should arise naturally in such physical applications.

In this paper we introduce quasi-Hopf \ast -algebras (\ast -quasi-Hopf algebras, \ast -QHA). Our definition is motivated by the twisting construction of Drinfeld [7] which turns a Hopf algebra H into a quasi-Hopf algebra. The twisting operation changes the co-algebra structure of H in such a way that the twisted co-product is no longer co-associative. The algebra structure of H is not affected by twisting. The axioms introduced by Drinfeld for quasi-Hopf algebras ensure that any Hopf algebra will be twisted into a quasi-Hopf algebra and that the class of quasi-Hopf algebras is itself closed under the twisting operation. This larger class of algebras contains the Hopf algebras within it since every Hopf algebra is trivially a quasi-Hopf algebra.

A Hopf algebra H may be equipped with a \ast -operation $\dagger : H \rightarrow H$, whenever the base field \mathbb{F} over which it is defined admits a conjugation operation. A Hopf \ast -algebra [28] is a Hopf algebra equipped with a \ast -operation $\dagger : H \rightarrow H$ such that on the algebra part of H , the conjugation \dagger obeys the usual axioms of a \ast -algebra, and such that on the co-algebra part, the co-product $\Delta : H \rightarrow H \otimes H$ and the co-unit $\varepsilon : H \rightarrow \mathbb{F}$ are \ast -algebra homomorphisms. The antipode S of a Hopf \ast -algebra necessarily obeys $S(a)^\dagger = S^{-1}(a^\dagger), \forall a \in H$. This is a direct consequence of the uniqueness of the antipode.

Since a Hopf \ast -algebra is a Hopf algebra, twisting changes the Hopf algebra part into a quasi-Hopf algebra. As twisting does not affect the algebra structure of H , the \ast -algebra part of H is unchanged. The \ast -structure on the co-algebra H is twisted in such a way that the twisted co-product Δ_F is no longer a \ast -algebra homomorphism. Nonetheless, Δ_F is a \ast -algebra homomorphism up to conjugation by the self adjoint twist $\Omega = (FF^\dagger)^{-1}$

$$\Delta_F(a)^\dagger = \Omega \Delta_F(a^\dagger) \Omega^{-1}, \quad \forall a \in H.$$

Twisting makes H into a quasi-Hopf algebra and thus it has a co-associator Φ_F (induced by F). We show that the co-associator Φ_F is related to its conjugate inverse $(\Phi_F^\dagger)^{-1}$ by the same twist Ω i.e.

$$(\Phi_F^\dagger)^{-1} = (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi_F (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1}).$$

Our definition of quasi-Hopf \ast -algebras is motivated by these observations. We define a \ast -quasi-Hopf algebra to be a quasi-Hopf algebra equipped with a conjugation $\dagger : H \rightarrow H$ and a twist $\Omega \in H \otimes H$ such that

$$\begin{aligned} \varepsilon(a^\dagger) &= \overline{\varepsilon(a)}, \quad \forall a \in H \\ \Delta(a)^\dagger &= \Omega \Delta(a^\dagger) \Omega^{-1}, \quad \forall a \in H \\ (\Phi^\dagger)^{-1} &= (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1}) \equiv \Phi_\Omega. \end{aligned}$$

This definition ensures that any Hopf $*$ -algebra H is twisted into a quasi-Hopf $*$ -algebra. We show that the class of quasi-Hopf $*$ -algebras is closed under twisting. Unlike the Hopf algebra case, the antipode S of a quasi-Hopf algebra is not unique. In the quasi-Hopf $*$ -algebra setting this means that S is not forced to satisfy any particular condition.

We develop the general theory of quasi-Hopf $*$ -algebras and investigate the relationship between the antipode S and the conjugation operation \dagger on H . The effect of the Drinfeld twist on the $*$ -canonical element Ω is determined and an explicit expression for the conjugate of the Drinfeld twist is derived. Quasi-triangular quasi-Hopf $*$ -algebras are introduced. As in the Hopf algebra case, there are two natural classes of quasi-triangular quasi-Hopf $*$ -algebras. In the type I case, the R -matrix satisfies $(\mathcal{R}^\dagger)^{-1} = \Omega^T \mathcal{R} \Omega^{-1}$, whilst for the type II case it satisfies $(\mathcal{R}^\dagger)^{-1} = \Omega^T (\mathcal{R}^T)^{-1} \Omega^{-1}$. These reduce to the antireal and real cases of Majid [20], respectively in the Hopf algebra case where $\Omega = 1 \otimes 1$.

A further motivation for our definition comes from the quantised universal enveloping algebra $U_q(L)$ of a semi-simple Lie algebra L , when $q \in \mathbb{C}$ is a complex phase. For q real and positive $U_q(L)$ is a Hopf $*$ -algebra. However, when $q \in \mathbb{C}$ is a complex phase, $\bar{q} = q^{-1}$ on conjugation, so that the conjugate of the co-product has the natural structure of the opposite co-algebra i.e.

$$\Delta(a)^\dagger = \Delta^T(a^\dagger), \forall a \in U_q(L).$$

Thus when q is a phase, H is not a Hopf $*$ -algebra as noted in [19]. Since $U_q(L)$ is quasitriangular, it has an R -matrix \mathcal{R} . Now \mathcal{R} is a twist and satisfies $\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}$ so that

$$\Delta(a)^\dagger = \mathcal{R}\Delta(a^\dagger)\mathcal{R}^{-1}.$$

We take $U_q(L)$ to be a quasi-Hopf algebra with trivial co-associator $\Phi = 1$. Now, $\Phi_{\mathcal{R}} = 1 \otimes 1 \otimes 1$ follows from the quantum Yang-Baxter equation, so that the $(\Phi^\dagger)^{-1} = \Phi_{\mathcal{R}}$ is trivially satisfied. Thus $U_q(L)$ for q a phase has the structure of a $*$ -quasi-Hopf algebra with $*$ -canonical element \mathcal{R} .

2 Preliminaries

We begin by recalling the definitions and basic properties of quasi-bialgebras (QBA) and quasi-Hopf algebras (QHA).

Definition 1. A quasi-bialgebra H is a unital associative algebra over a field \mathbb{F} , equipped with algebra homomorphisms $\varepsilon : H \rightarrow \mathbb{F}$ (co-unit), $\Delta : H \rightarrow H \otimes H$ (co-product) and an invertible element $\Phi \in H \otimes H \otimes H$ (co-associator), satisfying

$$(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta \quad (2.1)$$

$$(1 \otimes \Delta)\Delta(a) = \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \quad \forall a \in H \quad (2.2)$$

$$(\Delta \otimes 1 \otimes 1)\Phi (1 \otimes 1 \otimes \Delta)\Phi = (\Phi \otimes 1) (1 \otimes \Delta \otimes 1)\Phi (1 \otimes \Phi) \quad (2.3)$$

$$(1 \otimes \varepsilon \otimes 1)\Phi = 1. \quad (2.4)$$

A quasi-bialgebra H equipped with an algebra anti-homomorphism $S : H \rightarrow H$ (antipode) and canonical elements $\alpha, \beta \in H$ satisfying

$$\sum_{\nu} S(X_{\nu})\alpha Y_{\nu}\beta S(Z_{\nu}) = 1 = \sum_{\nu} \bar{X}_{\nu}\beta S(\bar{Y}_{\nu})\alpha \bar{Z}_{\nu} \quad (2.5)$$

$$\sum_{(a)} S(a_{(1)})\alpha a_{(2)} = \varepsilon(a)\alpha, \quad \sum_{(a)} a_{(1)}\beta S(a_{(2)}) = \varepsilon(a)\beta, \quad \forall a \in H. \quad (2.6)$$

is called a quasi-Hopf algebra.

Above we have used Sweedler's [27] notation for the co-product

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \quad \forall a \in H.$$

The co-product is no longer co-associative for QHA necessitating an extension to Sweedler's notation

$$\begin{aligned} (1 \otimes \Delta)\Delta(a) &= \sum_{(a)} a_{(1)} \otimes \Delta(a_{(2)}) = \sum_{(a)} a_{(1)} \otimes a_{(2)}^{(1)} \otimes a_{(2)}^{(2)} \\ (\Delta \otimes 1)\Delta(a) &= \sum_{(a)} \Delta(a_{(1)}) \otimes a_{(2)} = \sum_{(a)} a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes a_{(2)}. \end{aligned}$$

For the co-associator we follow the notation of [12, 13, 14] and write

$$\Phi = \sum_{\nu} X_{\nu} \otimes Y_{\nu} \otimes Z_{\nu}, \quad \Phi^{-1} = \sum_{\nu} \bar{X}_{\nu} \otimes \bar{Y}_{\nu} \otimes \bar{Z}_{\nu}.$$

We adopt the above notation throughout and in general omit the summation sign from expressions, with the convention that repeated indices are to be summed over.

It follows from equations (2.1), (2.3) and (2.4) that the co-associator Φ has the following useful properties

$$(\varepsilon \otimes 1 \otimes 1)\Phi = 1 = (1 \otimes 1 \otimes \varepsilon)\Phi.$$

Throughout we assume bijectivity of the antipode S so that S^{-1} exists. The antipode equations (2.5), (2.6) imply

$$\begin{aligned} \varepsilon(\alpha)\varepsilon(\beta) &= 1 \\ \varepsilon(S(a)) &= \varepsilon(S^{-1}(a)) = \varepsilon(a), \quad \forall a \in H. \end{aligned}$$

Let H be a QHA, an element $F \in H \otimes H$ is called a twist (or gauge transformation) if it is invertible and satisfies the co-unit property

$$(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1. \quad (2.7)$$

The operation of twisting operation allows one to construct a new QHA H_F from H , called the twisted structure induced by F , with the same antipode and co-unit, but with co-product, co-associator and canonical elements given by

$$\Delta_F(a) = F\Delta(a)F^{-1}, \quad \forall a \in H \quad (2.8)$$

$$\Phi_F = (F \otimes 1)(\Delta \otimes 1)F \Phi (1 \otimes \Delta)F^{-1}(1 \otimes F^{-1}) \quad (2.9)$$

$$\alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1}, \quad \beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F. \quad (2.10)$$

Above $m : H \otimes H \rightarrow H$ is the multiplication map $m \cdot (a \otimes b) = ab$.

Let $T : H \otimes H \rightarrow H \otimes H$ be the usual twist map $T(a \otimes b) = b \otimes a$. Recall that a quasi-Hopf algebra H is also a quasi-Hopf algebra with the opposite co-product $T \cdot \Delta$ as follows,

Proposition 1. *Let H be a QHA. Then the opposite QHA, H^{cop} is a QHA with co-product $\Delta^T = T \cdot \Delta$, co-associator $\Phi^T \equiv \Phi_{321}^{-1}$, antipode S^{-1} and canonical elements $\alpha^T \equiv S^{-1}(\alpha)$, $\beta^T \equiv S^{-1}(\beta)$.*

3 Twisting on Hopf \ast -algebras

In this and the following sections we take the base field to be the field of complex numbers \mathbb{C} . Recall that a bi-algebra is a QBA with trivial co-associator $\Phi = 1 \otimes 1 \otimes 1$. Similarly a Hopf algebra is a QHA with trivial co-associator and trivial canonical elements $\alpha = \beta = 1$.

Definition 2. A bi-algebra H is called a \ast -bi-algebra if it admits an antilinear map $\dagger : H \rightarrow H$ (conjugation operation) satisfying

$$(a^\dagger)^\dagger = a \quad (3.11)$$

$$(ab)^\dagger = b^\dagger a^\dagger \quad (3.12)$$

$$\varepsilon(a^\dagger) = \overline{\varepsilon(a)} \quad (3.13)$$

$$\Delta(a^\dagger) = \Delta(a)^\dagger, \quad \forall a, b \in H \quad (3.14)$$

where \dagger extends to a conjugation operation on all of $H \otimes H$ in a natural way so that

$$(a \otimes b)^\dagger = a^\dagger \otimes b^\dagger, \quad \forall a, b \in H.$$

Equations (3.11) and (3.12) are equivalent to the usual definition of a conjugation operation (also referred to as a \ast -operation) on the algebra H , whilst equations (3.13) and (3.14) are the compatibility conditions with the coalgebra structure: i.e. they determine \ast -algebra homomorphisms. In (3.13) the overbar denotes complex conjugation over \mathbb{C} : we adopt this convention throughout.

A \ast -bi-algebra H which admits an antipode S is called a Hopf \ast -algebra. For a Hopf \ast -algebra we necessarily have for the antipode S [26, 28]

Lemma 1.

$$S(a)^\dagger = S^{-1}(a^\dagger).$$

Proof. This follows from the uniqueness of the antipode S (as the inverse of the identity map on H under the convolution product) i.e. $S : H \rightarrow H$ is uniquely defined by

$$S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \varepsilon(a), \quad \forall a \in H.$$

Now define $\tilde{S} : H \rightarrow H$ by

$$\tilde{S}(a) = [S^{-1}(a^\dagger)]^\dagger \equiv S^{-1}(a^\dagger)^\dagger.$$

Then, since \dagger is compatible with Δ ,

$$\begin{aligned} \tilde{S}(a_{(1)})a_{(2)} &= \{S^{-1}[a_{(1)}^\dagger S(a_{(2)}^\dagger)]\}^\dagger \\ &= [\overline{\varepsilon(a)}]^\dagger = \varepsilon(a) \end{aligned}$$

and similarly

$$a_{(1)}\tilde{S}(a_{(2)}) = \varepsilon(a), \quad \forall a \in H.$$

Thus by the uniqueness of S , $\tilde{S} = S$ which is sufficient to prove the result. \square

Remark. The square of the antipode, S^2 determines an algebra homomorphism, in fact an algebra automorphism, which from lemma 1 satisfies $S^2(a^\dagger) = S^{-2}(a)^\dagger$. Thus S^2 does not determine a \ast -algebra homomorphism.

For the QHA case the situation with the antipode S is more complicated in view of the fact, that the antipode S is no longer unique [7]. Thus Lemma 1 does not hold for QHA.

In order to formulate a suitable definition for \ast -QHA we investigate how twisting alters the \ast -structure of a Hopf \ast -algebra. Let H be a Hopf algebra and $F \in H \otimes H$ an arbitrary twist. The twisted structure induced by F on H is no longer a Hopf algebra but is instead a QHA. The twisted structure is obtained by setting $\Phi = 1 \otimes 1 \otimes 1$, $\alpha = \beta = 1$ into equations (2.8 - 2.10), giving

$$\begin{aligned}\Delta_F(a) &= F\Delta(a)F^{-1} \\ \Phi_F &= (F \otimes 1) \cdot (\Delta \otimes 1)F \cdot (1 \otimes \Delta)F^{-1} \cdot (1 \otimes F^{-1}) \\ \alpha_F &= m \cdot (S \otimes 1)F^{-1}, \quad \beta_F = m \cdot (1 \otimes S)F.\end{aligned}\tag{3.15}$$

The counit ε and antipode S are unchanged.

Note that $\tilde{\Delta}_F$ defined by

$$\tilde{\Delta}_F(a) = \Delta_F(a^\dagger)^\dagger$$

determines another co-product on H . It shall be shown below for the general case, that H is in fact a QHA with the above co-product, with co-associator $\tilde{\Phi}_F = (\Phi_F^\dagger)^{-1}$ and canonical elements $\tilde{\alpha} = S^{-1}(\beta_F)^\dagger$, $\tilde{\beta} = S^{-1}(\alpha_F)^\dagger$.

Since the co-product Δ is compatible with \dagger we have

$$\begin{aligned}\Delta_F(a)^\dagger &= [F\Delta(a)F^{-1}]^\dagger \\ &= (F^\dagger)^{-1}\Delta(a)^\dagger F^\dagger \\ &= (F^\dagger)^{-1}\Delta(a^\dagger)F^\dagger \\ &= \Omega\Delta_F(a^\dagger)\Omega^{-1}\end{aligned}$$

where $\Omega = (FF^\dagger)^{-1}$ is a self-adjoint twist. Thus

$$\tilde{\Delta}_F(a) = \Delta_F(a^\dagger)^\dagger = \Omega\Delta_F(a)\Omega^{-1}, \quad \forall a \in H$$

so $\tilde{\Delta}_F$ is obtained from Δ_F by twisting with a (self-adjoint) twist Ω , or equivalently $\Delta_F(a)^\dagger = \Omega\Delta_F(a^\dagger)\Omega^{-1}$ as above.

Similarly for the co-associator

$$\Phi_F = (F \otimes 1) (\Delta \otimes 1)F (1 \otimes \Delta)F^{-1} (1 \otimes F^{-1}),$$

since Δ is compatible with \dagger

$$\begin{aligned}(\Phi_F^\dagger)^{-1} &= (F^{\dagger^{-1}} \otimes 1) (\Delta \otimes 1)F^{\dagger^{-1}} (1 \otimes \Delta)F^\dagger (1 \otimes F^\dagger) \\ &= (F^{\dagger^{-1}} \otimes 1) (\Delta \otimes 1)F^{\dagger^{-1}} [(\Delta \otimes 1)F^{-1} (F^{-1} \otimes 1) \\ &\quad \Phi_F (1 \otimes F) (1 \otimes \Delta)F] (1 \otimes \Delta)F^\dagger (1 \otimes F^\dagger) \\ &= (\Omega \otimes 1) (\Delta_F \otimes 1)\Omega \Phi_F (1 \otimes \Delta_F)\Omega^{-1} (1 \otimes \Omega^{-1}) \\ &= (\Phi_F)_\Omega\end{aligned}$$

so that $(\Phi_F^\dagger)^{-1}$ is also obtained from Φ_F by twisting with the (self-adjoint) twist $\Omega = (FF^\dagger)^{-1}$ as above.

We are now in a position to introduce the primary object of our investigation.

4 *-Quasi-Hopf algebras

Definition 3. A QHA H is called a *-QHA if it admits a conjugation operation \dagger and a twist $\Omega \in H \otimes H$, called the *-canonical element, satisfying

$$\varepsilon(a^\dagger) = \overline{\varepsilon(a)}, \quad \forall a \in H \quad (4.16)$$

$$\Delta(a)^\dagger = \Omega \Delta(a^\dagger) \Omega^{-1}, \quad \forall a \in H \quad (4.17)$$

$$(\Phi^\dagger)^{-1} = \Phi_\Omega = (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1}). \quad (4.18)$$

Our definition is motivated by the observation that any QHA obtained by twisting from a Hopf *-algebra is a quasi-Hopf *-algebra (*-QHA).

Following the previous section we define a new co-product $\tilde{\Delta}$ on H by

$$\tilde{\Delta}(a) = \Delta(a^\dagger)^\dagger, \quad \forall a \in H. \quad (4.19)$$

With this co-product H also determines a *-QHA, as will be seen below. In view of the previous section we may have imposed the extra conditions $\Omega^\dagger = \Omega$ (i.e. Ω is self-adjoint) and $S(a)^\dagger = S^{-1}(a^\dagger), \forall a \in H$ but we will not do this below. However, we define

Definition 4. If $\Omega = \Omega^\dagger$ we call a *-QHA H self-conjugate. If the antipode S satisfies

$$S(a)^\dagger = S^{-1}(a^\dagger), \quad \forall a \in H$$

we say that S is *-compatible.

In general for a *-QHA, the antipode S is not *-compatible, however we shall see that S is almost *-compatible.

Equations (4.17,4.18) impose strong conditions on the *-canonical element Ω . Indeed,

$$\begin{aligned} \Delta(a) &= [\Delta(a^\dagger)^\dagger]^\dagger \\ &\stackrel{(4.17)}{=} [\Omega \Delta(a^\dagger) \Omega^{-1}]^\dagger \\ &= (\Omega^{-1})^\dagger \Delta(a^\dagger)^\dagger \Omega^\dagger \\ &\stackrel{(4.17)}{=} (\Omega^{-1})^\dagger \Omega \Delta(a) \Omega^{-1} \Omega^\dagger, \quad \forall a \in H \end{aligned} \quad (4.20)$$

so that $\Omega^{-1} \Omega^\dagger$, and its inverse, must commute with the co-product Δ . We say that Ω is *quasi-self adjoint*.

We thus have, from equation (4.17),

$$\tilde{\Delta}(a) = \Delta(a^\dagger)^\dagger = \Omega \Delta(a) \Omega^{-1} \stackrel{(4.20)}{=} \Omega^\dagger \Delta(a) (\Omega^\dagger)^{-1} \quad (4.20')$$

or equivalently

$$\Delta(a)^\dagger = \Delta_\Omega(a^\dagger) = \Delta_{\Omega^\dagger}(a^\dagger), \quad \forall a \in H.$$

Thus we might expect that Ω^\dagger is also a *-canonical element for H . This is indeed the case.

Proposition 2. Ω^\dagger is also a *-canonical element for H called the conjugate *-canonical element.

Proof. It remains to check (4.18). To this end we have

$$(\Phi^\dagger)^{-1} = \Phi_\Omega = (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1})$$

Taking the conjugate inverse of this equation (i.e. apply \dagger followed by the inverse) and noting from (4.20') that $\dagger \cdot \Delta = \tilde{\Delta} \cdot \dagger$ gives

$$\begin{aligned} \Phi &= (\Omega^{\dagger^{-1}} \otimes 1)(\tilde{\Delta} \otimes 1)\Omega^{\dagger^{-1}} \Phi^{\dagger^{-1}} (1 \otimes \tilde{\Delta})\Omega^\dagger(1 \otimes \Omega^\dagger) \\ &\stackrel{(4.20')}{=} (\Delta \otimes 1)\Omega^{\dagger^{-1}}(\Omega^{\dagger^{-1}} \otimes 1) \Phi^{\dagger^{-1}}(1 \otimes \Omega^\dagger)(1 \otimes \Delta)\Omega^\dagger \end{aligned}$$

and hence

$$(\Phi^\dagger)^{-1} = (\Omega^\dagger \otimes 1)(\Delta \otimes 1)\Omega^\dagger \Phi (1 \otimes \Delta)\Omega^{\dagger^{-1}}(1 \otimes \Omega^{\dagger^{-1}}) = \Phi_{\Omega^\dagger}$$

which proves the result. \square

It follows that for a $*$ -QHA H the $*$ -canonical element Ω is not in general unique.

We now demonstrate that with the co-product $\tilde{\Delta}$ of equation (4.19), H is also a $*$ -QHA. In fact we have

Proposition 3. *Suppose H is any QHA admitting a conjugation operation $\dagger : H \rightarrow H$ satisfying only eq. (4.16). Then H is a QHA with the same co-unit ε but with co-product $\tilde{\Delta}$, co-associator $\tilde{\Phi} = (\Phi^\dagger)^{-1}$, canonical elements $\tilde{\alpha} = S^{-1}(\beta)^\dagger, \tilde{\beta} = S^{-1}(\alpha)^\dagger$ and antipode \tilde{S} defined by*

$$\tilde{S}(a) = (S^{-1}(a^\dagger))^\dagger, \quad \forall a \in H. \quad (4.21)$$

Moreover, if H is a $*$ -QHA then H is also a $*$ -QHA with this structure but with canonical element $\tilde{\Omega} = \Omega^{-1}$.

Proof. First it is obvious that $\tilde{\Delta}, \varepsilon$ determine a coalgebra structure on H and are algebra homomorphisms. As to the co-associator $\tilde{\Phi}$ we have by applying \dagger to (2.2),

$$(1 \otimes \tilde{\Delta})\tilde{\Delta}(a^\dagger) = (\Phi^\dagger)^{-1}(\tilde{\Delta} \otimes 1)\tilde{\Delta}(a^\dagger)\Phi^\dagger, \quad \forall a \in H$$

which proves (2.2). As to property (2.3), taking the conjugate inverse of (2.3) gives immediately

$$(\tilde{\Delta} \otimes 1 \otimes 1)(\Phi^\dagger)^{-1} (1 \otimes 1 \otimes \tilde{\Delta})(\Phi^\dagger)^{-1} = (\Phi^{\dagger^{-1}} \otimes 1) (1 \otimes \tilde{\Delta} \otimes 1)(\Phi^\dagger)^{-1} (1 \otimes \Phi^{\dagger^{-1}})$$

as required. Property (2.4) is obvious, so it remains to consider (2.5) and (2.6). As to the former, we set

$$\tilde{\Phi} = \tilde{X}_\nu \otimes \tilde{Y}_\nu \otimes \tilde{Z}_\nu = (\Phi^\dagger)^{-1} = \bar{X}_\nu^\dagger \otimes \bar{Y}_\nu^\dagger \otimes \bar{Z}_\nu^\dagger$$

which implies

$$\tilde{S}(\tilde{X}_\nu)\tilde{\alpha}\tilde{Y}_\nu\tilde{\beta}\tilde{S}(\tilde{Z}_\nu) = \{S^{-1}[\bar{X}_\nu\beta S(\bar{Y}_\nu)\alpha\bar{Z}_\nu]\}^\dagger \stackrel{(2.5)}{=} 1$$

and similarly setting

$$\tilde{\Phi}^{-1} = \bar{\bar{X}}_\nu \otimes \bar{\bar{Y}}_\nu \otimes \bar{\bar{Z}}_\nu = \Phi^\dagger = X_\nu^\dagger \otimes Y_\nu^\dagger \otimes Z_\nu^\dagger$$

we have

$$\bar{\bar{X}}_\nu\tilde{\beta}\tilde{S}(\bar{\bar{Y}}_\nu)\tilde{\alpha}\bar{\bar{Z}}_\nu = \{S^{-1}[S(X_\nu)\alpha Y_\nu\beta S(Z_\nu)]\}^\dagger \stackrel{(2.5)}{=} 1.$$

As to property (2.6) we have

$$\tilde{\Delta}(a) = \Delta(a^\dagger)^\dagger = (a^\dagger_{(1)})^\dagger \otimes (a^\dagger_{(2)})^\dagger$$

so that

$$\begin{aligned} \tilde{S}[(a^\dagger_{(1)})^\dagger] \tilde{\alpha}(a^\dagger_{(2)})^\dagger &= S^{-1}[(a^\dagger_{(1)}) \beta S(a^\dagger_{(2)})]^\dagger \\ &= \varepsilon(a) S^{-1}(\beta)^\dagger = \varepsilon(a) \tilde{\alpha} \end{aligned}$$

and similarly for $\tilde{\beta}$ as required. This proves that H gives rise to a QHA under the given structure.

Finally if H is a $*$ -QHA with $*$ -canonical element Ω then H is also a $*$ -QHA under the above structure but with $*$ -canonical element Ω^{-1} . To see this we have from equation (4.20'),

$$\tilde{\Delta}(a)^\dagger = \Delta(a^\dagger) = \Omega^{-1} \Delta(a)^\dagger \Omega = \Omega^{-1} \tilde{\Delta}(a^\dagger) \Omega, \quad \forall a \in H$$

which proves (4.17), while for (4.18) we have

$$\tilde{\Phi}_{\Omega^{-1}} = (\Phi^\dagger)_{\Omega^{-1}}^{-1} = (\Phi_\Omega)_{\Omega^{-1}} = \Phi_{\Omega^{-1}\Omega} = \Phi = (\tilde{\Phi})^\dagger{}^{-1}$$

so that Ω^{-1} is a $*$ -canonical element for this structure thus making it a $*$ -QHA. \square

When H admits a conjugation operation \dagger satisfying (4.16) it ensures that with the structure of proposition 3, H is also a QHA. Conditions (4.17, 4.18) are equivalent to this QHA structure being obtainable, up to equivalence modulo (S, α, β) , by twisting with Ω .

We now demonstrate that the category of $*$ -QHAs is invariant under twisting, as is the subcategory of self-conjugate $*$ -QHAs. This latter observation is important as it demonstrates that we cannot obtain a self-conjugate $*$ -QHA from a non-self-conjugate one by twisting.

Theorem 1. *Let H be a (self-conjugate) $*$ -QHA with $*$ -canonical element Ω and $F \in H \otimes H$ an arbitrary twist. Then H is also a (self-conjugate) $*$ -QHA with the twisted structure of equations (2.8) with $*$ -canonical element $\Omega_F = (F^\dagger)^{-1} \Omega F^{-1}$. Moreover if the antipode S is $*$ -compatible then it is $*$ -compatible under this twisted structure.*

Proof. It suffices to prove (4.17, 4.18). For the twisted co-product we have

$$\begin{aligned} \Delta_F(a)^\dagger &= [F \Delta(a) F^{-1}]^\dagger = (F^{-1})^\dagger \Delta(a)^\dagger F^\dagger \\ &\stackrel{(4.17)}{=} (F^{-1})^\dagger \Omega \Delta(a^\dagger) \Omega^{-1} F^\dagger \\ &= \Omega_F \Delta_F(a^\dagger) \Omega_F^{-1} \end{aligned}$$

with $\Omega_F = (F^{-1})^\dagger \Omega F^{-1}$ as stated. As to the co-associator we have

$$\Phi_F = (F \otimes 1) (\Delta \otimes 1) F \Phi (1 \otimes \Delta) F^{-1} (1 \otimes F^{-1})$$

so that taking the conjugate inverse gives

$$(\Phi_F^\dagger)^{-1} = (F^{\dagger^{-1}} \otimes 1) (\tilde{\Delta} \otimes 1) F^{\dagger^{-1}} \Phi^{\dagger^{-1}} (1 \otimes \tilde{\Delta}) F^\dagger (1 \otimes F^\dagger)$$

with $\tilde{\Delta}$ as in equation (4.19) [also cf equation (4.20')] and where

$$(\Phi^\dagger)^{-1} \stackrel{(4.17)}{=} \Phi_\Omega = (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1}).$$

Thus by equation (4.20')

$$\begin{aligned}
(\Phi_F^\dagger)^{-1} &= (F^{\dagger^{-1}} \otimes 1) (\Delta_\Omega \otimes 1) F^{\dagger^{-1}} (\Omega \otimes 1) (\Delta \otimes 1) \Omega \\
&\quad \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1}) (1 \otimes \Delta_\Omega) F^\dagger (1 \otimes F^\dagger) \\
&= (F^{\dagger^{-1}} \Omega \otimes 1) (\Delta \otimes 1) (F^{\dagger^{-1}} \Omega) \\
&\quad \Phi (1 \otimes \Delta) (\Omega^{-1} F^\dagger) (1 \otimes \Omega^{-1} F^\dagger) \\
&= (F^{\dagger^{-1}} \Omega \otimes 1) (\Delta \otimes 1) (F^{\dagger^{-1}} \Omega) (\Delta \otimes 1) F^{-1} (F^{-1} \otimes 1) \\
&\quad \Phi_F (1 \otimes F) (1 \otimes \Delta) F (1 \otimes \Delta) (\Omega^{-1} F^\dagger) (1 \otimes \Omega^{-1} F^\dagger) \\
&= (\Omega_F \otimes 1) (\Delta_F \otimes 1) \Omega_F \Phi_F (1 \otimes \Delta_F) \Omega_F^{-1} (1 \otimes \Omega_F^{-1}) \\
&= (\Phi_F)_{\Omega_F}
\end{aligned} \tag{4.22}$$

with $\Omega_F = (F^\dagger)^{-1} \Omega F^{-1}$ as required. Thus under the twisted structure induced by F , H is a $*$ -QHA with $*$ -canonical element Ω_F as stated. If moreover H is self conjugate, so that Ω is self adjoint, so too is Ω_F which implies H is also a self-conjugate $*$ -QHA under the twisted structure. Finally $*$ -compatibility of the antipode S is obviously twist invariant since S remains unchanged under twisting. \square

We refer to the twisted structure above as the twisted $*$ -QHA induced by F . The above result has a number of interesting consequences to which we now turn.

Proposition 4. *Let H be a $*$ -QHA with $*$ -canonical element Ω . Then H is also a $*$ -QHA under the opposite structure of proposition 1 with $*$ -canonical element $\Omega^T = T \cdot \Omega$.*

Proof. Recall that H is a QHA under the opposite structure with co-product $\Delta^T = T \cdot \Delta$, co-associator $\Phi^T = \Phi_{321}^{-1}$ and antipode S^{-1} , with the same co-unit. To prove this gives rise to a $*$ -QHA it suffices to prove (4.17, 4.18). For the co-product we have,

$$\Delta^T(a)^\dagger = T \cdot [\Delta(a)^\dagger] = T \cdot [\Omega \Delta(a^\dagger) \Omega^{-1}] = \Omega^T \Delta^T(a^\dagger) (\Omega^T)^{-1}, \quad \forall a \in H$$

as required. For the co-associator we have

$$(\Phi^T)^\dagger^{-1} = \Phi_{321}^\dagger. \tag{4.23}$$

Now since

$$(\Phi^\dagger)^{-1} = \Phi_\Omega = (\Omega \otimes 1) (\Delta \otimes 1) \Omega \Phi (1 \otimes \Delta) \Omega^{-1} (1 \otimes \Omega^{-1})$$

we have

$$\Phi^\dagger = (1 \otimes \Omega) (1 \otimes \Delta) \Omega \Phi^{-1} (\Delta \otimes 1) \Omega^{-1} (\Omega^{-1} \otimes 1)$$

and hence

$$\begin{aligned}
(\Phi^T)^\dagger^{-1} &\stackrel{(4.23)}{=} [(1 \otimes \Omega) (1 \otimes \Delta) \Omega \Phi^{-1} (\Delta \otimes 1) \Omega^{-1} (\Omega^{-1} \otimes 1)]_{321} \\
&= (\Omega^T \otimes 1) (\Delta^T \otimes 1) \Omega^T \Phi^T (1 \otimes \Delta^T) \Omega^{T^{-1}} (1 \otimes \Omega^{T^{-1}}) = (\Phi^T)_{\Omega^T}
\end{aligned}$$

so $(\Phi^T)^\dagger^{-1}$ is obtained from Φ^T by twisting with Ω^T under the opposite structure. Thus H is also a $*$ -QHA with $*$ -canonical element Ω^T under the opposite structure as required. If moreover H is self-conjugate, so that Ω is self-adjoint, so too is Ω^T . Thus under the opposite structure, a self-conjugate $*$ -QHA, is also self-conjugate. Obviously if the antipode S of H is $*$ -compatible so too is the antipode S^{-1} for the opposite structure. \square

We have already seen that the $*$ -canonical element Ω for a $*$ -QHA is not unique, since Ω^\dagger also gives rise to a $*$ -canonical element. We thus conclude this section with the following observation on the uniqueness, and existence of $*$ -canonical elements.

Let $F, G \in H \otimes H$ be twists on H . The composite twist FG is given by first twisting with G and then twisting H_G by F so that

$$X_{FG} = (X_G)_F$$

where X is one of $\Delta, \Phi, \alpha, \beta, \mathcal{R}$. A twist $C \in H \otimes H$ which preserve the QBA structure on H , so that

$$\begin{aligned}\Delta_C(a) &= \Delta(a), \quad \forall a \in H \\ \Phi_C &= \Phi\end{aligned}$$

is called a *compatible* twist [12]. The set of compatible twists is a subgroup of the group of all twists on H .

Theorem 2. *Let H be a $*$ -QHA with $*$ -canonical element Ω . Then $\Gamma \in H \otimes H$ is also a $*$ -canonical element for H if and only if there exists a (unique) compatible twist $F \in H \otimes H$ such that $\Gamma = \Omega F$.*

Proof. Follows from a direct computation using the composition laws for twists. \square

Corollary. *For a $*$ -QHA H , there is a one to one correspondence between $*$ -canonical elements and compatible twists on H .*

In particular there must exist a compatible twist $C \in H \otimes H$ such that $\Omega^\dagger = \Omega C$. Thus we see that Ω is almost self-adjoint, hence the term quasi-self adjoint. As will be seen below the explicit choice of $*$ -canonical element Ω has no effect on the algebraic properties of $*$ -QHAs, due to the special nature of compatible twists.

The existence of a conjugation operation on a $*$ -QHA and the properties (4.16- 4.18) imply some interrelationships between \dagger and the algebraic structure of H to which we now turn.

5 Compatibility of $*$ and algebra properties

A QHA differs from a Hopf algebra in that the antipode S and its corresponding canonical elements α, β are not unique. Nevertheless, the antipode and its corresponding canonical elements are almost unique as the following result due to Drinfeld [7] shows.

Theorem 3. *Suppose H is also a QHA with antipode \tilde{S} and canonical elements $\tilde{\alpha}, \tilde{\beta}$. Then there exists a unique invertible $v \in H$ such that*

$$v\alpha = \tilde{\alpha}, \quad \tilde{\beta}v = \beta, \quad \tilde{S}(a) = vS(a)v^{-1}, \quad \forall a \in H.$$

Explicitly

$$\begin{aligned}(i) \quad v &= \tilde{S}(X_\nu)\tilde{\alpha}Y_\nu\beta S(Z_\nu) = \tilde{S}(S^{-1}(\bar{X}_\nu))\tilde{S}(S^{-1}(\beta))\tilde{S}(\bar{Y}_\nu)\tilde{\alpha}\bar{Z}_\nu \\ (ii) \quad v^{-1} &= S(X_\nu)\alpha Y_\nu\tilde{\beta}\tilde{S}(Z_\nu) = \bar{X}_\nu\tilde{\beta}\tilde{S}(\bar{Y}_\nu)\tilde{S}(S^{-1}(\alpha))\tilde{S}(S^{-1}(\bar{Z}_\nu))\end{aligned}$$

For arbitrary invertible $v \in H$, the triple $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ defined by

$$\tilde{S}(a) = vS(a)v^{-1}, \quad \tilde{\alpha} = v\alpha, \quad \tilde{\beta} = \beta v^{-1}$$

satisfies equations (2.5), (2.6) and hence gives rise to an antipode \tilde{S} with corresponding canonical elements $\tilde{\alpha}, \tilde{\beta}$. There is thus a 1 – 1 correspondence between triples $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ and invertible $v \in H$. We say that these structures are equivalent (modulo (S, α, β)) as they give rise to equivalent QHA structures.

Proposition 3 shows that H is a $*$ -QHA with co-unit ε , co-product $\tilde{\Delta} = \Delta_\Omega = \Delta_{\Omega^\dagger}$, co-associator $(\Phi^\dagger)^{-1} = \Phi_\Omega = \Phi_{\Omega^\dagger}$, canonical elements $\tilde{\alpha} = S^{-1}(\beta)^\dagger$, $\tilde{\beta} = S^{-1}(\alpha)^\dagger$ and with antipode \tilde{S} given by equation (4.21). On the other hand, from equations (4.16-4.18), H is also a $*$ -QHA under the twisted structure induced by Ω (or Ω^\dagger) with the same co-unit, co-product and co-associator but with antipode S and twisted canonical elements given by equation (2.10)

$$\alpha_\Omega = m \cdot (1 \otimes \alpha)(S \otimes 1)\Omega^{-1}, \quad \beta_\Omega = m \cdot (1 \otimes \beta)(1 \otimes S)\Omega$$

or

$$\alpha_{\Omega^\dagger} = m \cdot (1 \otimes \alpha)(S \otimes 1)(\Omega^\dagger)^{-1}, \quad \beta_{\Omega^\dagger} = m \cdot (1 \otimes \beta)(1 \otimes S)\Omega^\dagger.$$

Hence these structures must be equivalent. We have immediately from Theorem 3

Proposition 5. *There exists a unique invertible $w \in H$ such that*

$$\begin{aligned} (i) \quad & wS^{-1}(\beta)^\dagger = \alpha_\Omega, \quad \beta_\Omega w = S^{-1}(\alpha)^\dagger \\ (ii) \quad & S(a) = w\tilde{S}(a)w^{-1}, \quad \forall a \in H. \end{aligned} \tag{5.24}$$

Explicitly

$$\begin{aligned} w &= S(\bar{X}_\nu^\dagger)\alpha_\Omega\bar{Y}_\nu^\dagger S^{-1}(\alpha)^\dagger\tilde{S}(\bar{Z}_\nu^\dagger) \\ &= S(\tilde{S}^{-1}(X_\nu^\dagger))S(\tilde{S}^{-1}[S^{-1}(\alpha)^\dagger])S(Y_\nu^\dagger)\alpha_\Omega Z_\nu^\dagger \\ w^{-1} &= \tilde{S}(\bar{X}_\nu^\dagger)S^{-1}(\beta)^\dagger\bar{Y}_\nu^\dagger\beta_\Omega S(\bar{Z}_\nu^\dagger) \\ &= X_\nu^\dagger\beta_\Omega S(Y_\nu^\dagger)S(\tilde{S}^{-1}[S^{-1}(\beta)^\dagger])S(\tilde{S}^{-1}(Z_\nu^\dagger)). \end{aligned}$$

Above we used the fact that the co-associator for the QHA we are considering is $(\Phi^\dagger)^{-1}$ together with the antipode \tilde{S} and canonical elements $S^{-1}(\beta)^\dagger, S^{-1}(\alpha)^\dagger$ respectively. We then applied Theorem 3 to this structure with $(\tilde{S}, \tilde{\alpha}, \tilde{\beta}) \equiv (S, \alpha_\Omega, \beta_\Omega)$.

Corollary 1.

$$\begin{aligned} w^\dagger &= S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu\alpha_\Omega^\dagger\tilde{S}^{-1}(\bar{X}_\nu) \\ (w^{-1})^\dagger &= \tilde{S}^{-1}(\bar{Z}_\nu)\beta_\Omega^\dagger\bar{Y}_\nu S^{-1}(\beta)S^{-1}(\bar{X}_\nu). \end{aligned}$$

Corollary 2. *S is $*$ -compatible i.e. $S(a)^\dagger = S^{-1}(a^\dagger)$ or equivalently $\tilde{S}(a) \equiv [S^{-1}(a^\dagger)]^\dagger = S(a)$, $\forall a \in H$ if and only if w as above is a central element.*

The results above, particularly equation (5.24)(ii) and Corollary 2 might be thought to depend on the $*$ -canonical element Ω . To see this is not the case, let Γ be another $*$ -canonical element so $\Gamma = \Omega C$, for some compatible twist $C \in H \otimes H$. The corresponding twisted canonical elements are

$$\alpha_\Gamma = \alpha_{\Omega C} = (\alpha_C)_\Omega, \quad \beta_\Gamma = \beta_{\Omega C} = (\beta_C)_\Omega.$$

From Theorem 3 there exists a unique invertible element $z \in H$ such that

$$\alpha_C = z\alpha, \quad \beta_C = z^{-1}\beta$$

with

$$S(a) = zS(a)z^{-1}, \quad \forall a \in H.$$

The element z is thus central. Now,

$$\alpha_\Gamma = z\alpha_\Omega, \quad \beta_\Gamma = z^{-1}\beta_\Omega.$$

The corresponding w -operator, given by replacing $\alpha_\Omega, \beta_\Omega$ with $\alpha_\Gamma, \beta_\Gamma$ respectively, is thus given by

$$w_\Gamma = zw, \quad w_\Gamma^{-1} = z^{-1}w^{-1}$$

so that, in particular

$$S(a) = w\tilde{S}(a)w^{-1} = w_\Gamma\tilde{S}(a)w_\Gamma^{-1}$$

the latter equality holding identically. Thus the results of Proposition 5 and its corollaries are independent (modulo an invertible central element) of the canonical element chosen. \square

We have shown previously [12] that the v operator of Theorem 3 is universal i.e. unchanged under twisting by an arbitrary twist F , so that for any operator v arising from the application of Theorem 3 we have $v_F = v$. Since the w operator arises precisely in this way, it follows that

Theorem 4. *The operator w is universal, i.e. twist invariant.*

Remark. The universality of w can also be shown by direct calculation. The operator w can be expressed in the following simpler form,

$$w = S(X_\nu)\alpha Y_\nu S^{-1}(\alpha_{\Omega^\dagger})^\dagger \tilde{S}(Z_\nu). \quad (5.25)$$

The results of Proposition 5 have a number of interesting consequences which we summarise below:

Lemma 2. *(Notation as above)*

- (i) $\tilde{S}(S^{-1}(a)) = w^{-1}aw, \quad \forall a \in H$
- (ii) $S(\tilde{S}^{-1}(a)) = waw^{-1}, \quad \forall a \in H$
- (iii) $S(\tilde{S}^{-1}(w)) = \tilde{S}(S^{-1}(w)) = w$

so that

$$S^{-1}(w)^\dagger = S(w^\dagger) \quad \text{or} \quad S^{-1}(w) = \tilde{S}^{-1}(w) \quad (5.26)$$

and similarly for w^{-1} .

- (iv) $S^{-1}(a) = S^{-1}(w)\tilde{S}^{-1}(a)S^{-1}(w^{-1})$
 $= \tilde{S}^{-1}(w)\tilde{S}^{-1}(a)\tilde{S}^{-1}(w^{-1}), \quad \forall a \in H.$

If $z \in H$ is a central element then

$$(v) \quad S^{-1}(z) = \tilde{S}^{-1}(z)$$

so that

$$S^{-1}(z)^\dagger = S(z^\dagger).$$

Proof. (i), (ii) and (v) follow directly from equation (5.24)(ii). Part (iii) is a direct consequence of (i) and (ii). Applying S^{-1} to part (ii) gives (iv). \square

Since Ω^\dagger is also a $*$ -canonical element for H we may replace Ω with Ω^\dagger in proposition 5 to give

Proposition 5' *There exists a unique invertible $\bar{w} \in H$ such that*

$$\begin{aligned} (i) \quad & \bar{w}S^{-1}(\beta)^\dagger = \alpha_{\Omega^\dagger}, \quad \beta_{\Omega^\dagger}\bar{w} = S^{-1}(\alpha)^\dagger \\ (ii) \quad & S(a) = \bar{w}\tilde{S}(a)\bar{w}^{-1}, \quad \forall a \in H. \end{aligned}$$

Explicitly \bar{w} is given as in proposition 5 with Ω replaced by Ω^\dagger .

Corollary. $c = w^{-1}\bar{w} = \bar{w}w^{-1}$ is a central element with inverse

$$c^{-1} = w\bar{w}^{-1} = \bar{w}^{-1}w. \quad (5.27)$$

Thus the results of lemma 2 also hold for \bar{w} . In view of the definition (4.21) of \tilde{S} ; i.e.

$$\tilde{S}(a) = S^{-1}(a^\dagger)^\dagger, \quad \forall a \in H$$

the canonical elements of proposition 3 may be written

$$S^{-1}(\beta)^\dagger = \tilde{S}(\beta^\dagger), \quad S^{-1}(\alpha)^\dagger = \tilde{S}(\alpha^\dagger).$$

Also, by taking the Hermitian conjugate of (5.24)(ii), equation (5.24) may be written as

$$\begin{aligned} (i) \quad & w\tilde{S}(\beta^\dagger) = \alpha_\Omega, \quad \beta_\Omega w = \tilde{S}(\alpha^\dagger) \\ (ii) \quad & S(a)^\dagger = (w^\dagger)^{-1}S^{-1}(a^\dagger)w^\dagger, \quad \forall a \in H \end{aligned} \quad (5.24')$$

and similarly for Ω^\dagger , with w replaced by \bar{w} .

It might be thought that the operators w, \bar{w} of propositions (5,5') respectively, are directly related. This turns out to be the case. We first need

Lemma 3. *(notation as above):*

$$\begin{aligned} (i) \quad & \alpha_\Omega^\dagger = S^{-1}(\beta)S^{-1}(\bar{w}), \quad \beta_\Omega^\dagger = S^{-1}(\bar{w}^{-1})S^{-1}(\alpha) \\ (ii) \quad & \alpha_{\Omega^\dagger}^\dagger = S^{-1}(\beta)S^{-1}(w), \quad \beta_{\Omega^\dagger}^\dagger = S^{-1}(w^{-1})S^{-1}(\alpha). \end{aligned}$$

Proof. By symmetry, it suffices to prove (i). Below we write (summation over repeated indices assumed)

$$\Omega = \Omega_i \otimes \Omega^i, \quad \Omega^{-1} = \bar{\Omega}_i \otimes \bar{\Omega}^i.$$

We have

$$\alpha_\Omega^\dagger = [S(\bar{\Omega}_i)\alpha\bar{\Omega}^i]^\dagger = (\bar{\Omega}^i)^\dagger\alpha^\dagger\tilde{S}^{-1}(\bar{\Omega}_i^\dagger)$$

where

$$\alpha^\dagger \stackrel{(5.24')(i)}{=} \tilde{S}^{-1}(\beta_{\Omega^\dagger}\bar{w}) = \tilde{S}^{-1}(\bar{w})\tilde{S}^{-1}(\beta_{\Omega^\dagger}) \stackrel{(5.26)(iii)}{=} S^{-1}(\bar{w})\tilde{S}^{-1}(\beta_{\Omega^\dagger})$$

and we have used the fact that eq. (5.24') also holds for Ω^\dagger with w replaced by \bar{w} . Thus

$$\begin{aligned} \alpha_\Omega^\dagger &= (\bar{\Omega}^i)^\dagger S^{-1}(\bar{w})\tilde{S}^{-1}(\beta_{\Omega^\dagger})\tilde{S}^{-1}(\bar{\Omega}_i^\dagger) \\ &\stackrel{(5.26)(iv)}{=} (\bar{\Omega}^i)^\dagger S^{-1}(\beta_{\Omega^\dagger})S^{-1}(\bar{\Omega}_i^\dagger)S^{-1}(\bar{w}) \\ &= S^{-1}(\beta)S^{-1}(\bar{w}). \end{aligned}$$

The result for β_Ω^\dagger is proved in a similar way. \square

We are now in a position to compute w^\dagger . We have

Proposition 6. (*notation as above*)

$$w^\dagger = S^{-1}(\bar{w}), \quad \bar{w}^\dagger = S^{-1}(w).$$

In particular, for the central element c of equation (5.27), we have

$$c^\dagger = S^{-1}(c^{-1}).$$

Proof. Using corollary 1 to proposition 5 we have

$$\begin{aligned} w^\dagger &= S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu\alpha_\Omega^\dagger\tilde{S}^{-1}(\bar{X}_\nu) \\ &\stackrel{\text{lemma(3)(i)}}{=} S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu S^{-1}(\beta)S^{-1}(\bar{w})\tilde{S}^{-1}(\bar{X}_\nu) \\ &\stackrel{(5.26)(iv)}{=} S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu S^{-1}(\beta)S^{-1}(\bar{X}_\nu)S^{-1}(\bar{w}) \\ &= S^{-1}[\bar{X}_\nu\beta S(\bar{Y}_\nu)\alpha\bar{Z}_\nu]S^{-1}(\bar{w}) \stackrel{(2.5)}{=} S^{-1}(\bar{w}). \end{aligned}$$

Thus

$$\bar{w} = S(w^\dagger)$$

which implies

$$\bar{w}^\dagger = S(w^\dagger)^\dagger = \tilde{S}^{-1}(w) \stackrel{(5.26)(iii)}{=} S^{-1}(w)$$

and similarly

$$w^\dagger = S^{-1}(\bar{w}).$$

Finally, as to the central element $c = w^{-1}\bar{w}$ of equation (5.27) we have, from the above

$$c^\dagger = \bar{w}^\dagger(w^\dagger)^{-1} = S^{-1}(w)S^{-1}(\bar{w}^{-1}) = S^{-1}(\bar{w}^{-1}w) = S^{-1}(c^{-1})$$

which proves the result. \square

Corollary.

$$\begin{aligned} \bar{w} &= S(w^\dagger) = S(\tilde{S}^{-1}(\bar{X}_\nu))S(\alpha_\Omega^\dagger)S(\bar{Y}_\nu)\alpha\bar{Z}_\nu \\ \bar{w}^{-1} &= S(w^\dagger)^{-1} = \bar{X}_\nu\beta S(\bar{Y}_\nu)S(\beta_\Omega^\dagger)S(\tilde{S}^{-1}(\bar{Z}_\nu)) \end{aligned}$$

and similarly for w, w^{-1} with Ω replaced by Ω^\dagger .

Proof. Follows from applying the result above and S to Corollary 1 of Proposition 5. \square

In the case S is $*$ -compatible so that $S = \tilde{S}$, or equivalently w, \bar{w} are both central, the corollary above reduces to

$$\begin{aligned} \bar{w} &= \bar{X}_\nu S(\alpha_\Omega^\dagger)S(\bar{Y}_\nu)\alpha\bar{Z}_\nu \\ \bar{w}^{-1} &= \bar{X}_\nu\beta S(\bar{Y}_\nu)S(\beta_\Omega^\dagger)\bar{Z}_\nu \end{aligned}$$

and similarly for w, w^{-1} with Ω replaced by Ω^\dagger . This gives a useful expansion directly in terms of Φ^{-1} .

In the case the $*$ -QHA H is self-conjugate, so that $\Omega = \Omega^\dagger$ and $w = \bar{w}$, the result of proposition 6 gives

$$w^\dagger = S^{-1}(w)$$

while the central element c of equation (5.27) is obviously trivial.

We conclude this section with a simple observation concerning conjugation of the twisted operators of equations (2.8), of use below.

Lemma 4. *Let $F \in H \otimes H$ be a twist on a $*$ -QHA H with $*$ -canonical element Ω . Then*

$$\begin{aligned} (i) \quad & \Delta_F^\dagger(a) = \Delta_{(F^\dagger)^{-1}\Omega}(a^\dagger), \quad \forall a \in H \\ (ii) \quad & (\Phi_F^\dagger)^{-1} = \Phi_{(F^\dagger)^{-1}\Omega}, \quad (\Phi_F^\dagger) = \Phi_{(F^\dagger)^{-1}\Omega}^{-1} \\ (iii) \quad & \alpha_F^\dagger = S^{-1}[\beta_{(F^\dagger)^{-1}\Omega}]S^{-1}(w), \quad \beta_F^\dagger = S^{-1}(w^{-1})S^{-1}[\alpha_{(F^\dagger)^{-1}\Omega}]. \end{aligned}$$

Proof. (i)

$$\begin{aligned} \Delta_F(a)^\dagger &= [F\Delta(a)F^{-1}]^\dagger = (F^{-1})^\dagger \Delta(a)^\dagger F^\dagger \\ &= (F^{-1})^\dagger \Omega \Delta(a^\dagger) \Omega^{-1} F^\dagger \\ &= \Delta_{(F^\dagger)^{-1}\Omega}(a^\dagger), \quad \forall a \in H. \end{aligned}$$

(ii) From equation (4.22) in the proof of Theorem 1, we have

$$\begin{aligned} (\Phi_F^\dagger)^{-1} &= (F^\dagger)^{-1} \Omega \otimes 1 \cdot (\Delta \otimes 1)(F^\dagger)^{-1} \Omega \cdot \Phi \cdot (1 \otimes \Delta)(\Omega^{-1} F^\dagger) \cdot (1 \otimes \Omega^{-1} F^\dagger) \\ &= \Phi_{(F^\dagger)^{-1}\Omega}. \end{aligned}$$

(iii) First set (summation over repeated indices)

$$F = f_i \otimes f^i, \quad F^{-1} = \bar{f}_i \otimes \bar{f}^i.$$

Then

$$\begin{aligned} \alpha_F^\dagger &= [S(\bar{f}_i) \alpha \bar{f}^i]^\dagger = (\bar{f}^i)^\dagger \alpha^\dagger \tilde{S}^{-1}(\bar{f}_i^\dagger) \\ &\stackrel{(5.24')(i)}{=} (\bar{f}^i)^\dagger \tilde{S}^{-1}(\beta_\Omega w) \tilde{S}^{-1}(\bar{f}_i^\dagger) \\ &\stackrel{(5.26)(iii)}{=} (\bar{f}^i)^\dagger S^{-1}(w) \tilde{S}^{-1}(\beta_\Omega) \tilde{S}^{-1}(\bar{f}_i^\dagger) \\ &\stackrel{(5.26)(iv)}{=} (\bar{f}^i)^\dagger S^{-1}(\beta_\Omega) S^{-1}(\bar{f}_i^\dagger) S^{-1}(w) \\ &= S^{-1}[\beta_{(F^\dagger)^{-1}\Omega}] S^{-1}(w). \end{aligned}$$

The proof for β_F^\dagger is similar. □

Remark. In the case $F = \Omega^\dagger$, the results of Lemma (4)(iii) reduce to those of Lemma (3)(iii,iv).

We now turn our attention to some more advanced results on the compatibility of the algebraic and $*$ -properties of $*$ -QHAs.

6 Conjugation of the Drinfeld Twist

Observe that Δ' defined by

$$\Delta'(a) = (S \otimes S) \Delta^T(S^{-1}(a)), \quad \forall a \in H \tag{6.28}$$

also determines a co-product on H .

Proposition 7. *Let H be a QHA, then H is also a QHA with the same co-unit ε and antipode S but with co-product Δ' , co-associator $\Phi' = (S \otimes S \otimes S) \Phi_{321}$ and canonical elements $\alpha' = S(\beta)$, $\beta' = S(\alpha)$.*

Drinfeld has proved that this QHA structure is obtained by twisting with the Drinfeld twist, herein denoted F_δ , given explicitly by

$$(i) \quad F_\delta = (S \otimes S) \Delta^T(X_\nu) \cdot \gamma \cdot \Delta(Y_\nu \beta S(Z_\nu)) \\ = \Delta'(\bar{X}_\nu \beta S(\bar{Y}_\nu)) \cdot \gamma \cdot \Delta(\bar{Z}_\nu)$$

where

$$(ii) \quad \gamma = S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i$$

with

$$(iii) \quad A_i \otimes B_i \otimes C_i \otimes D_i = \begin{cases} (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi \\ \text{or} \\ (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta) \Phi^{-1} . \end{cases} \quad (6.29)$$

The inverse of F_δ is given explicitly by

$$(i) \quad F_\delta^{-1} = \Delta(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(\bar{Y}_\nu) \alpha \bar{Z}_\nu) \\ = \Delta(S(X_\nu) \alpha Y_\nu) \cdot \bar{\gamma} \cdot (S \otimes S) \Delta^T(Z_\nu)$$

where

$$(ii) \quad \bar{\gamma} = \bar{A}_i \beta S(\bar{D}_i) \otimes \bar{B}_i \beta S(\bar{C}_i)$$

with

$$(iii) \quad \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i = \begin{cases} (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \\ \text{or} \\ (1 \otimes 1 \otimes \Delta) \Phi \cdot (1 \otimes \Phi^{-1}) . \end{cases} \quad (6.30)$$

Replacing S with S^{-1} in eq. (6.28) we obtain yet another co-product Δ_0 on H :

$$\Delta_0(a) = (S^{-1} \otimes S^{-1}) \Delta^T(S(a)), \quad \forall a \in H. \quad (6.28')$$

We have the following analogue of proposition 7, the proof of which parallels that of [14] proposition 4, but with S and S^{-1} interchanged:

Proposition 7' *H is also a QHA with the same co-unit ε and antipode S but with co-product Δ_0 , co-associator $\Phi_0 = (S^{-1} \otimes S^{-1} \otimes S^{-1}) \Phi_{321}$ and canonical elements $\alpha_0 = S^{-1}(\beta)$, $\beta_0 = S^{-1}(\alpha)$ respectively.*

By symmetry we would expect this structure to be obtainable twisting. Indeed we have

Theorem 5. : *The QHA structure of proposition 7' is obtained by twisting with*

$$F_0 \equiv (S^{-1} \otimes S^{-1}) F_\delta^T \quad (6.31)$$

herein referred to as the second Drinfeld twist, where F_δ is the Drinfeld twist and $F_\delta^T = T \cdot F_\delta$.

Throughout we assume that H is a $*$ -QHA with $*$ -canonical element

$$\Omega = \Omega_i \otimes \Omega^i, \quad \Omega^{-1} = \bar{\Omega}_i \otimes \bar{\Omega}^i$$

(summation over repeated indices). In view of Theorem 1, H is also a $*$ -QHA under the QHA structures of propositions 7, 7' induced by twisting with the Drinfeld twists F_δ and F_0 respectively, with F_δ as in equation (6.29) and F_0 as in equation (6.31). Further from Theorem 1, the $*$ -canonical elements for these QHAs are given by

$$\Omega' = (F_\delta^\dagger)^{-1} \Omega F_\delta^{-1}, \quad \Omega_0 = (F_0^\dagger)^{-1} \Omega F_0^{-1}. \quad (6.32)$$

It is one of the aims below to obtain the operators of equation (6.32) explicitly in terms of F_δ and Ω .

First it is worth noting, with Δ' as in equation (6.28),

$$\begin{aligned} [\Delta'(a)]^\dagger &= [(S \otimes S) \Delta^T(S^{-1}(a))]^\dagger = (\tilde{S}^{-1} \otimes \tilde{S}^{-1}) \cdot T \cdot [\Delta(S^{-1}(a))^\dagger] \\ &= (\tilde{S}^{-1} \otimes \tilde{S}^{-1}) \cdot T \cdot [\Omega \Delta(S^{-1}(a)^\dagger) \Omega^{-1}] \\ &= (\tilde{S}^{-1} \otimes \tilde{S}^{-1}) [\Omega^T \Delta^T(\tilde{S}(a^\dagger)) (\Omega^T)^{-1}]. \end{aligned}$$

Now using (5.24)(ii) and (5.26)(iv) respectively, we may write

$$\tilde{S}(a) = w^{-1} S(a) w, \quad \tilde{S}^{-1}(a) = S^{-1}(w^{-1}) S^{-1}(a) S^{-1}(w)$$

so that

$$\begin{aligned} [\Delta'(a)]^\dagger &= W^{-1} (S^{-1} \otimes S^{-1}) [\Omega^T \Delta^T(w^{-1} S(a^\dagger) w) (\Omega^T)^{-1}] W \\ &= W^{-1} (S^{-1} \otimes S^{-1}) (\Omega^T)^{-1} \Delta_0(S^{-1}(w) a^\dagger S^{-1}(w^{-1})) (S^{-1} \otimes S^{-1}) \Omega^T W \end{aligned}$$

where we have introduced the following operators

$$W = S^{-1}(w) \otimes S^{-1}(w), \quad W^{-1} = S^{-1}(w^{-1}) \otimes S^{-1}(w^{-1})$$

in order to simplify the notation. Thus, with F_0 as in equation (6.31),

$$[\Delta'(a)]^\dagger = W^{-1} (S^{-1} \otimes S^{-1}) (\Omega^T)^{-1} \cdot F_0 \Delta(S^{-1}(w)) \Delta(a^\dagger) \Delta(S^{-1}(w^{-1})) F_0^{-1} (S^{-1} \otimes S^{-1}) \Omega^T W. \quad (6.33)$$

On the other hand we have

$$[\Delta'(a)]^\dagger = [F_\delta \Delta(a) F_\delta^{-1}]^\dagger = (F_\delta^{-1})^\dagger \Delta(a)^\dagger F_\delta^\dagger = (F_\delta^{-1})^\dagger \Omega \Delta(a^\dagger) \Omega^{-1} F_\delta^\dagger.$$

By comparison with equation (6.33), it follows that the operator

$$\Omega^{-1} F_\delta^\dagger W^{-1} (S^{-1} \otimes S^{-1}) (\Omega^T)^{-1} F_0 \Delta(S^{-1}(w)) \quad (6.34)$$

must commute with the co-product Δ . Below we show in fact that equation (6.34) reduces to $1 \otimes 1$.

It is first useful to determine the behaviour of $\bar{\gamma}$ in equation (6.30)(ii) under an arbitrary twist $G \in H \otimes H$. Under the twisted structure induced by G the operator $\bar{\gamma}$ is twisted to $\bar{\gamma}_G$, given by equation (6.30)(ii,iii) for the twisted structure, so that

$$\begin{aligned} (i) \quad \bar{\gamma}_G &= \bar{A}_i^G \beta_G S(\bar{D}_i^G) \otimes \bar{B}_i^G \beta_G S(\bar{C}_i^G) \\ \text{where } (ii) \quad \bar{A}_i^G \otimes \bar{B}_i^G \otimes \bar{C}_i^G \otimes \bar{D}_i^G &= (\Delta_G \otimes 1 \otimes 1) \Phi_G^{-1} \cdot (\Phi_G \otimes 1). \end{aligned} \quad (6.35)$$

We have shown in a previous publication [12] that

$$\bar{\gamma}_G = G \Delta(g_i) \bar{\gamma} (S \otimes S) (G^T \Delta^T(g^i)). \quad (6.36)$$

Proposition 8. *Let γ be the operator of equation (6.29)(ii). Then*

$$\begin{aligned}\gamma^\dagger &= \Omega\Delta(\Omega^i) (S^{-1} \otimes S^{-1})\bar{\gamma}^T (S^{-1} \otimes S^{-1})[\Omega^T \Delta^T(\Omega_i)] W \\ &= \Omega\Delta(\Omega^i) (S^{-1} \otimes S^{-1})\bar{\gamma}^T (S^{-1} \otimes S^{-1})\Delta^T(\Omega_i) (S^{-1} \otimes S^{-1})\Omega^T W.\end{aligned}$$

Proof. From equation (6.29)(ii) we have

$$\begin{aligned}\gamma^\dagger &= C_i^\dagger \alpha^\dagger \tilde{S}^{-1}(B_i^\dagger) \otimes D_i^\dagger \alpha^\dagger \tilde{S}^{-1}(A_i^\dagger) \\ &\stackrel{(5.24')(ii)}{=} C_i^\dagger \tilde{S}^{-1}(w) \tilde{S}^{-1}(\beta_\Omega) \tilde{S}^{-1}(B_i^\dagger) \otimes D_i^\dagger \tilde{S}^{-1}(w) \tilde{S}^{-1}(\beta_\Omega) \tilde{S}^{-1}(A_i^\dagger) \\ &\stackrel{(5.26)(iii)}{=} C_i^\dagger S^{-1}(w) \tilde{S}^{-1}(\beta_\Omega) \tilde{S}^{-1}(B_i^\dagger) \otimes D_i^\dagger S^{-1}(w) \tilde{S}^{-1}(\beta_\Omega) \tilde{S}^{-1}(A_i^\dagger) \\ &\stackrel{(5.26)(iv)}{=} [C_i^\dagger S^{-1}(\beta_\Omega) S^{-1}(B_i^\dagger) \otimes D_i^\dagger S^{-1}(\beta_\Omega) S^{-1}(A_i^\dagger)] W.\end{aligned}$$

Now from equation (6.29)(iii)

$$\begin{aligned}A_i^\dagger \otimes B_i^\dagger \otimes C_i^\dagger \otimes D_i^\dagger &= [(\Phi^{-1} \otimes 1)(\Delta \otimes 1 \otimes 1)\Phi]^\dagger \\ &= (\Delta_\Omega \otimes 1 \otimes 1)\Phi^\dagger[(\Phi^\dagger)^{-1} \otimes 1] \\ &= (\Delta_\Omega \otimes 1 \otimes 1)\Phi_\Omega^{-1}(\Phi_\Omega \otimes 1) \\ &\stackrel{(6.30)(iii)}{=} \bar{A}_i^\Omega \otimes \bar{B}_i^\Omega \otimes \bar{C}_i^\Omega \otimes \bar{D}_i^\Omega\end{aligned}$$

which is the operator of equation (6.30)(iii) for the twisted structure induced by Ω (see equation (6.35)(ii)). Thus

$$\begin{aligned}\gamma^\dagger &= [\bar{C}_i^\Omega S^{-1}(\beta_\Omega) S^{-1}(\bar{B}_i^\Omega) \otimes \bar{D}_i^\Omega S^{-1}(\beta_\Omega) S^{-1}(\bar{A}_i^\Omega)] W \\ &= (S^{-1} \otimes S^{-1})[\bar{B}_i^\Omega \beta_\Omega S(\bar{C}_i^\Omega) \otimes \bar{A}_i^\Omega \beta_\Omega S(\bar{D}_i^\Omega)] W \\ &\stackrel{(6.35)(i)}{=} (S^{-1} \otimes S^{-1})(\bar{\gamma}_\Omega^T) W \\ &\stackrel{(6.36)}{=} (S^{-1} \otimes S^{-1}) \cdot T \cdot [\Omega\Delta(\Omega_i) \bar{\gamma} (S \otimes S)(\Omega^T \Delta^T(\Omega^i))] W \\ &= \Omega\Delta(\Omega_i) (S^{-1} \otimes S^{-1})\bar{\gamma}^T (S^{-1} \otimes S^{-1})(\Omega^T \Delta^T(\Omega^i)) W\end{aligned}\tag{6.37}$$

where, as usual $\bar{\gamma}^T \equiv T \cdot \bar{\gamma}$. This proves the result. \square

We are now in a position to compute F_δ^\dagger . From equation (6.29)(i) we have immediately

$$\begin{aligned}F_\delta^\dagger &= \Delta(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\{\Delta^T(S^{-1}[\bar{X}_\nu \beta S(\bar{Y}_\nu)])^\dagger\} \\ &= \Delta(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T([S^{-1}(\bar{X}_\nu \beta S(\bar{Y}_\nu))]^\dagger) \\ &= \Delta_\Omega(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[\tilde{S}[\tilde{S}^{-1}(\bar{Y}_\nu^\dagger)\beta^\dagger \bar{X}_\nu^\dagger]] \\ &= \Delta_\Omega(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[\tilde{S}(\bar{X}_\nu^\dagger)\tilde{S}(\beta^\dagger)\bar{Y}_\nu^\dagger] \\ &\stackrel{(5.24')(i)}{=} \Delta_\Omega(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[\tilde{S}(\bar{X}_\nu^\dagger)w^{-1}\alpha_\Omega \bar{Y}_\nu^\dagger] \\ &\stackrel{(5.24)(ii)}{=} \Delta_\Omega(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[w^{-1}S(\bar{X}_\nu^\dagger)\alpha_\Omega \bar{Y}_\nu^\dagger] \\ &= \Delta_\Omega(\bar{Z}_\nu)^\dagger \gamma^\dagger (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[S(\bar{X}_\nu^\dagger)\alpha_\Omega \bar{Y}_\nu^\dagger] (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T(w^{-1}) \\ &\stackrel{(6.37)}{=} \Delta_\Omega(\bar{Z}_\nu)^\dagger (S^{-1} \otimes S^{-1})\bar{\gamma}_\Omega^T W (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[S(\bar{X}_\nu^\dagger)\alpha_\Omega \bar{Y}_\nu^\dagger] (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T(w^{-1}) \\ &\stackrel{(5.26)(iv)}{=} \Delta_\Omega(\bar{Z}_\nu)^\dagger (S^{-1} \otimes S^{-1})\bar{\gamma}_\Omega^T (\tilde{S}^{-1} \otimes \tilde{S}^{-1})\Delta_\Omega^T[S(\bar{X}_\nu^\dagger)\alpha_\Omega \bar{Y}_\nu^\dagger] (S^{-1} \otimes S^{-1})\Delta_\Omega^T(w^{-1}) W \\ &= (S^{-1} \otimes S^{-1}) \cdot T \cdot [\Delta_\Omega(S(\bar{X}_\nu^\dagger)\alpha_\Omega \bar{Y}_\nu^\dagger) \bar{\gamma}_\Omega (S \otimes S)\Delta_\Omega^T(\bar{Z}_\nu^\dagger)] (S^{-1} \otimes S^{-1})\Delta_\Omega^T(w^{-1}) W.\end{aligned}$$

Now we may write

$$\bar{X}_\nu^\dagger \otimes \bar{Y}_\nu^\dagger \otimes \bar{Z}_\nu^\dagger = (\Phi^\dagger)^{-1} = \Phi_\Omega = X_\nu^\Omega \otimes y_\nu^\Omega \otimes Z_\nu^\Omega$$

which is the co-associator for the twisted structure induced by Ω . Thus we have,

$$\begin{aligned} F_\delta^\dagger &= (S^{-1} \otimes S^{-1}) \cdot T \cdot [\Delta_\Omega(S(X_\nu^\Omega) \alpha_\Omega Y_\nu^\Omega) \bar{\gamma}_\Omega (S \otimes S) \Delta_\Omega^T(Z_\nu^\Omega)] (S^{-1} \otimes S^{-1}) \Delta_\Omega^T(w^{-1}) W \\ &\stackrel{(6.30)(ii)}{=} (S^{-1} \otimes S^{-1}) \cdot T \cdot [(F_\delta^\Omega)^{-1}] (S^{-1} \otimes S^{-1}) \Delta_\Omega^T(w^{-1}) W \end{aligned}$$

where F_δ^Ω is the Drinfeld twist for the twisted structure induced by Ω and $(F_\delta^\Omega)^{-1}$ is its inverse.

We now make use of the following theorem proved in [12].

Theorem 6. *Let $G \in H \otimes H$ be a twist on a QHA H . Then under the twisted structure induced by G , F_δ^{-1} is twisted to*

$$(F_\delta^G)^{-1} \equiv (F_\delta^{-1})_G = G \cdot F_\delta^{-1} \cdot (S \otimes S) G^T.$$

It follows from Theorem 6 that

$$\begin{aligned} F_\delta^\dagger &= (S^{-1} \otimes S^{-1}) \cdot T \cdot [\Omega F_\delta^{-1} (S \otimes S) \Omega^T] (S^{-1} \otimes S^{-1}) \Delta_\Omega^T(w^{-1}) W \\ &= \Omega (S^{-1} \otimes S^{-1}) (F_\delta^T)^{-1} (S^{-1} \otimes S^{-1}) \Omega^T (S^{-1} \otimes S^{-1}) \Delta_\Omega^T(w^{-1}) W \\ &\stackrel{(6.31)}{=} \Omega F_0^{-1} (S^{-1} \otimes S^{-1}) \Delta^T(w^{-1}) (S^{-1} \otimes S^{-1}) \Omega^T W \\ &= \Omega F_0^{-1} \Delta_0 (S^{-1}(w^{-1})) (S^{-1} \otimes S^{-1}) \Omega^T W \end{aligned}$$

with Δ_0 the co-product of equation (6.28'). We thus arrive at our main result

Theorem 7.

$$F_\delta^\dagger = \Omega \Delta (S^{-1}(w^{-1})) F_0^{-1} (S^{-1} \otimes S^{-1}) \Omega^T W.$$

Corollary 1. *With F_0 as in equation (6.31),*

$$F_0^\dagger = \Omega \Delta (w^{-1}) F_\delta^{-1} (S \otimes S) \Omega^T (w \otimes w).$$

Proof. From equation (6.31), $F_0 = (S^{-1} \otimes S^{-1}) F_\delta^T$, which implies

$$F_0^\dagger = (\tilde{S} \otimes \tilde{S}) [(F_\delta^T)^\dagger].$$

Now from Theorem 7 we get

$$\begin{aligned} F_0^\dagger &= (\tilde{S} \otimes \tilde{S}) [\Omega^T \Delta^T (S^{-1}(w^{-1})) (F_0^T)^{-1}] (S^{-1} \otimes S^{-1}) \Omega W \\ &= (\tilde{S} \otimes \tilde{S}) W (\tilde{S} \otimes \tilde{S}) [\Omega^T \Delta^T (S^{-1}(w^{-1})) (F_0^T)^{-1}] (S^{-1} \otimes S^{-1}) \Omega \\ &\stackrel{(5.26)(iii)}{=} (w \otimes w) (\tilde{S} \otimes \tilde{S}) [\Omega^T \Delta^T (S^{-1}(w^{-1})) (F_0^T)^{-1}] (S^{-1} \otimes S^{-1}) \Omega \\ &\stackrel{(5.24)(ii)}{=} (S \otimes S) [\Omega^T \Delta^T (S^{-1}(w^{-1})) (F_0^T)^{-1}] (S^{-1} \otimes S^{-1}) \Omega (w \otimes w). \end{aligned}$$

Now from equation (6.31), $(F_0^T)^{-1} = (S^{-1} \otimes S^{-1}) (F_\delta^{-1})$, so that

$$\begin{aligned} F_0^\dagger &= (S \otimes S) [\Omega^T \Delta^T (S^{-1}(w^{-1})) (S^{-1} \otimes S^{-1}) F_\delta^{-1} (S^{-1} \otimes S^{-1}) \Omega] (w \otimes w) \\ &= \Omega F_\delta^{-1} \Delta' (w^{-1}) (S \otimes S) \Omega^T (w \otimes w) \\ &= \Omega \Delta (w^{-1}) F_\delta^{-1} (S \otimes S) \Omega^T (w \otimes w) \end{aligned}$$

which proves the result. \square

Corollary 2. *The operator of equation (6.34) is given by*

$$\Omega^{-1}F_{\delta}^{\dagger}W(S^{-1} \otimes S^{-1})(\Omega^T)^{-1}F_0\Delta(S^{-1}(w)) = 1 \otimes 1.$$

Proof. Follows by an easy computation using Theorem 7. \square

If S is $*$ -compatible, so that $\tilde{S} = S$, the above result for F_{δ}^{\dagger} remains unaltered, except for the simplification that w is central.

The results above have a number of interesting consequences. In particular, we are now in a position to obtain the $*$ -canonical elements of equation (6.32) pertinent to the $*$ -QHAs of propositions 7, 7'. By a straightforward calculation using Theorem 7 and Corollary 1, we immediately obtain

Proposition 9.

$$\begin{aligned}\Omega' &\equiv (F_{\delta}^{\dagger})^{-1}\Omega F_{\delta}^{-1} \\ &= W(S^{-1} \otimes S^{-1})(\Omega^T)^{-1}F_0\Delta(S^{-1}(w))F_{\delta}^{-1} \\ &= [S^{-1}(w^{-1}) \otimes S^{-1}(w^{-1})](S^{-1} \otimes S^{-1})(\Omega^T)^{-1}F_0\Delta(S^{-1}(w))F_{\delta}^{-1} \\ \Omega_0 &\equiv (F_0^{\dagger})^{-1}\Omega F_0^{-1} \\ &= (w^{-1} \otimes w^{-1})(S \otimes S)(\Omega^T)^{-1}F_{\delta}\Delta(w)F_0^{-1}.\end{aligned}$$

All of the results of this section will obviously hold with Ω replaced by Ω^{\dagger} in which case w must be replaced by \bar{w} . In particular the result of proposition 8 will hold with Ω replaced by Ω^{\dagger} and w by \bar{w} . Taking the Hermitian conjugate of the resultant expression, using proposition 6, it is then easy to obtain an expression for $\bar{\gamma}^{\dagger}$ in terms of Ω, w and γ .

Replacing Ω, w with $\bar{\Omega}, \bar{w}$ respectively in proposition 9, we arrive at the corresponding conjugate $*$ -canonical elements

Proposition 9'

$$\begin{aligned}(\Omega')^{\dagger} &= (F_{\delta}^{\dagger})^{-1}\Omega^{\dagger}F_{\delta}^{-1} \\ &= [S^{-1}(\bar{w}^{-1}) \otimes S^{-1}(\bar{w}^{-1})](S^{-1} \otimes S^{-1})(\Omega^{\dagger})^{T-1}F_0\Delta(S^{-1}(\bar{w}))F_{\delta}^{-1} \\ \Omega_0^{\dagger} &= (F_0^{\dagger})^{-1}\Omega^{\dagger}F_0^{-1} \\ &= [\bar{w}^{-1} \otimes \bar{w}^{-1}](S \otimes S)(\Omega^{\dagger})^{T-1}F_{\delta}\Delta(\bar{w})F_0^{-1}.\end{aligned}$$

Note Replacing Ω with Ω^{\dagger} and w with \bar{w} in Corollary 1 to Theorem 7 gives

$$\begin{aligned}F_0^{\dagger} &= \Omega\Delta(w^{-1})F_{\delta}^{-1}(S \otimes S)\Omega^T(w \otimes w) \\ &= \Omega^{\dagger}\Delta(\bar{w}^{-1})F_{\delta}^{-1}(S \otimes S)(\Omega^T)^{\dagger}(\bar{w} \otimes \bar{w})\end{aligned}$$

which implies that

$$\Omega^{-1}\Omega^{\dagger}\Delta(\bar{w}^{-1})F_{\delta}^{-1}(S \otimes S)(\Omega^T)^{\dagger} = \Delta(w^{-1})F_{\delta}^{-1}(S \otimes S)\Omega^T(c^{-1} \otimes c^{-1})$$

where $c = w^{-1}\bar{w}$ is the central element of equation (5.27). Using the fact that $\Omega^{-1}\Omega^{\dagger}$ commutes with Δ , then gives

$$(\Omega^{-1}\Omega^{\dagger})F_{\delta}^{-1}(S \otimes S)(\Omega^T)^{\dagger}(S \otimes S)(\Omega^T)^{-1} = \Delta(c)F_{\delta}^{-1}(c^{-1} \otimes c^{-1})$$

or, to put it another way

$$(\Omega^{-1}\Omega^{\dagger})F_{\delta}^{-1}(S \otimes S)(\Omega^{-1}\Omega^{\dagger})^TF_{\delta} = (c^{-1} \otimes c^{-1})\Delta(c).$$

In the case H is self conjugate, so that $\Omega = \Omega^{\dagger}$, this just reduces to an identity.

We now consider the important case when the $*$ -QHA H is quasi-triangular.

7 The Quasi-triangular Case

A quasi-Hopf algebra H is called quasi-triangular if there exists an invertible element $\mathcal{R} \in H \otimes H$ called the R -matrix, such that

$$\Delta^T(a)\mathcal{R} = \mathcal{R}\Delta(a), \quad \forall a \in H \quad (7.38)$$

$$(\Delta \otimes 1)\mathcal{R} = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1} \quad (7.39)$$

$$(1 \otimes \Delta)\mathcal{R} = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123}. \quad (7.40)$$

Above, $\Delta^T(a) = T \cdot \Delta(a)$, where $T : H \otimes H \rightarrow H \otimes H$ is the usual twist map, $T(a \otimes b) = b \otimes a$. For the co-associator we have followed the conventions of [13, 14] so that,

$$\Phi_{231} = \sum_{\nu} Z_{\nu} \otimes Y_{\nu} \otimes X_{\nu}, \quad \Phi_{312}^{-1} = \sum_{\nu} \bar{Z}_{\nu} \otimes \bar{X}_{\nu} \otimes \bar{Y}_{\nu}, \quad \text{etcetera.}$$

We set

$$\mathcal{R} = \sum_i e_i \otimes e^i, \quad \mathcal{R}^{-1} = \sum_i \bar{e}_i \otimes \bar{e}^i$$

in terms of which

$$\mathcal{R}_{12} = \sum_i e_i \otimes e^i \otimes 1, \quad \mathcal{R}_{13} = \sum_i e_i \otimes 1 \otimes e^i, \quad \text{etcetera.}$$

Throughout this section we assume H is a \ast -QHA with \ast -canonical element Ω , which is moreover quasi-triangular, i.e. admits an R -matrix, $\mathcal{R} \in H \otimes H$ satisfying equations (7.38-7.40).

The R -matrix \mathcal{R} satisfies the additional relations

$$(\epsilon \otimes 1)\mathcal{R} = (1 \otimes \epsilon)\mathcal{R} = 1,$$

which follow from (7.39) and (7.40). Since \mathcal{R} is invertible and satisfies the co-unit property (2.7) it qualifies as a twist.

Recall that if H is a QTQHA then it is also a QTQHA under the opposite structure of proposition 1 but with opposite R -matrix $\mathcal{R}^T \equiv T \cdot \mathcal{R}$. Twisting H with the R -matrix \mathcal{R} gives rise to this opposite structure but with antipode S and canonical elements $\alpha_{\mathcal{R}}, \beta_{\mathcal{R}}$ given by equ. (2.10). Applying theorem 3 to these equivalent QHA structures gives

Theorem 8. *There exists a unique invertible $u \in H$ such that*

$$S(a) = uS^{-1}(a)u^{-1}, \quad \text{or } S^2(a) = uau^{-1}, \quad \forall a \in H$$

and

$$uS^{-1}(\alpha) = \alpha_{\mathcal{R}}, \quad \beta_{\mathcal{R}}u = S^{-1}(\beta). \quad (7.41)$$

Explicitly,

$$\begin{aligned} u &= S(Y_{\nu}\beta S(Z_{\nu}))\alpha_{\mathcal{R}}X_{\nu} = S(\bar{Z}_{\nu})\alpha_{\mathcal{R}}\bar{Y}_{\nu}S^{-1}(\beta)S^{-1}(\bar{X}_{\nu}) \\ u^{-1} &= Z_{\nu}\beta_{\mathcal{R}}S(S(X_{\nu})\alpha Y_{\nu}) = S^{-1}(\bar{Z}_{\nu})S^{-1}(\alpha)\bar{Y}_{\nu}\beta_{\mathcal{R}}S(\bar{X}_{\nu}). \end{aligned} \quad (7.42)$$

Now from the intertwining property (7.38) we have

$$\mathcal{R}\Delta(a^{\dagger}) = \Delta^T(a^{\dagger})\mathcal{R}$$

which gives, upon applying \dagger ,

$$\mathcal{R}^\dagger \tilde{\Delta}^T(a) = \tilde{\Delta}(a) \mathcal{R}^\dagger, \quad \text{or} \quad (\mathcal{R}^\dagger)^{-1} \tilde{\Delta}(a) = \tilde{\Delta}^T(a) (\mathcal{R}^\dagger)^{-1}, \quad \forall a \in H.$$

Thus $(\mathcal{R}^\dagger)^{-1}$ satisfies the intertwining property (7.38) for the co-product $\tilde{\Delta}$ of equation (4.19), given by [cf equation(4.20')]

$$\tilde{\Delta}(a) = \Delta(a^\dagger)^\dagger = \Delta_\Omega(a) = \Delta_{\Omega^\dagger}(a), \quad \forall a \in H.$$

We thus expect $(\mathcal{R}^\dagger)^{-1}$ to give rise to an R -matrix for H with the QHA structure of proposition 3, which is indeed the case.

Proposition 3' *Suppose H is any quasi-triangular QHA admitting a conjugation operation \dagger satisfying only equation (4.16). Then H is also a quasi-triangular QHA with the structure of proposition 3 with R -matrix $(\mathcal{R}^\dagger)^{-1}$.*

Proof. First recall that, with the structure of proposition 3, H is a QHA with the same co-unit but with co-product $\tilde{\Delta}$, co-associator $\tilde{\Phi} = (\Phi^\dagger)^{-1}$ and antipode \tilde{S} given by equation (4.21). We have already seen that if \mathcal{R} is an R -matrix for H then $(\mathcal{R}^\dagger)^{-1}$ satisfies the intertwining property (7.38) for this structure. It thus remains to consider (7.39, 7.40).

Taking the conjugate inverse of equations (7.39, 7.40) and using $\dagger \cdot \Delta = \tilde{\Delta} \cdot \dagger$, gives immediately

$$\begin{aligned} (\tilde{\Delta} \otimes 1)(\mathcal{R}^\dagger)^{-1} &= \Phi_{231}^\dagger (\mathcal{R}^\dagger)_{13}^{-1} (\Phi^\dagger)_{132}^{-1} (\mathcal{R}^\dagger)_{23}^{-1} \Phi_{123}^\dagger \\ (1 \otimes \tilde{\Delta})(\mathcal{R}^\dagger)^{-1} &= (\Phi^\dagger)_{312}^{-1} (\mathcal{R}^\dagger)_{13}^{-1} \Phi_{213}^\dagger (\mathcal{R}^\dagger)_{12}^{-1} (\Phi^\dagger)_{123}^{-1}. \end{aligned}$$

Setting $\tilde{\Phi} = (\Phi^\dagger)^{-1}$, which is the co-associator for this structure, implies

$$\begin{aligned} (\tilde{\Delta} \otimes 1)(\mathcal{R}^\dagger)^{-1} &= \tilde{\Phi}_{231}^{-1} (\mathcal{R}^\dagger)_{13}^{-1} \tilde{\Phi}_{132} (\mathcal{R}^\dagger)_{23}^{-1} \tilde{\Phi}_{123}^{-1} \\ (1 \otimes \tilde{\Delta})(\mathcal{R}^\dagger)^{-1} &= \tilde{\Phi}_{312}^{-1} (\mathcal{R}^\dagger)_{13}^{-1} \tilde{\Phi}_{213}^{-1} (\mathcal{R}^\dagger)_{12}^{-1} \tilde{\Phi}_{123} \end{aligned}$$

as required. \square

Corollary. *In the case H is a $*$ -QHA with $*$ -canonical element Ω ,*

$$\bar{\mathcal{R}} = (\Omega^T)^{-1} (\mathcal{R}^\dagger)^{-1} \Omega \quad (7.43)$$

determines an R -matrix for H

Proof. In such a case the QBA structure of proposition 3 is obtained by twisting with Ω (or Ω^\dagger); i.e. $\tilde{\Delta} = \Delta_\Omega$, $\tilde{\Phi} = \Phi_\Omega$. The result above shows that $(\mathcal{R}^\dagger)^{-1}$ is an R -matrix for this twisted structure. Since $\bar{\mathcal{R}}$ of equation (7.43) is obtained from $(\mathcal{R}^\dagger)^{-1}$ by twisting with Ω^{-1} it follows that $\bar{\mathcal{R}}$ must determine an R -matrix for H , i.e. satisfy equations (7.38 - 7.40), since Ω^{-1} will “undo” the twist Ω . \square

Remark. The above corollary may be proved directly.

Thus associated with the R -matrix $\bar{\mathcal{R}}$ of equation (7.43) we have a u -operator $\bar{u} = u_{\bar{\mathcal{R}}}$ and its inverse \bar{u}^{-1} given explicitly by equation (7.42) with \mathcal{R} replaced by $\bar{\mathcal{R}}$. Then $\bar{u} \in H$ is the unique operator satisfying

$$S^2(a) = \bar{u} a \bar{u}^{-1}, \quad \forall a \in H; \quad \bar{u} S^{-1}(\alpha) = \alpha_{\bar{\mathcal{R}}}, \quad \beta_{\bar{R}} \bar{u} = S^{-1}(\beta). \quad (7.44)$$

Here we explore the connection between u and \bar{u} . We first need, with \tilde{S} as in equation (4.21), the following

Lemma 5. (notation as in lemma 3)

$$\begin{aligned} (i) \quad & \tilde{S}^{-1}(\Omega_i)\beta_{\mathcal{R}}^{\dagger}\Omega^i = S^{-1}(w^{-1})S^{-1}(\alpha_{\mathcal{R}}) \\ (ii) \quad & \bar{\Omega}_i\alpha_{\mathcal{R}}^{\dagger}\tilde{S}^{-1}(\bar{\Omega}^i) = S^{-1}(\beta_{\bar{\mathcal{R}}})S^{-1}(w). \end{aligned}$$

Proof. (i) First note, from equation (7.43), that $(\mathcal{R}^{\dagger})^{-1} = \Omega^T \bar{\mathcal{R}} \Omega^{-1}$. Hence using lemma 4(iii) we have

$$\begin{aligned} \tilde{S}^{-1}(\Omega_i)\beta_{\mathcal{R}}^{\dagger}\Omega^i &= \tilde{S}^{-1}(\Omega_i)S^{-1}(w^{-1})S^{-1}[\alpha_{(\mathcal{R}^{\dagger})^{-1}\Omega}] \Omega^i \\ &= \tilde{S}^{-1}(\Omega_i)S^{-1}(w^{-1})S^{-1}[\alpha_{\Omega^T \bar{\mathcal{R}}}] \Omega^i \\ &\stackrel{(5.26)(iv)}{=} S^{-1}(w^{-1})S^{-1}(\Omega_i)S^{-1}[\alpha_{\Omega^T \bar{\mathcal{R}}}] \Omega^i \\ &= S^{-1}(w^{-1})S^{-1}[S(\Omega^i)\alpha_{\Omega^T \bar{\mathcal{R}}}\Omega_i] \\ &= S^{-1}(w^{-1})S^{-1}(\alpha_{\mathcal{R}}). \end{aligned}$$

Part (ii) is proved in a similar fashion. \square

We are now in a position to compute u^{\dagger} . From equation (7.42) we have immediately

$$u^{\dagger} = \tilde{S}(\bar{X}_{\nu}^{\dagger})S^{-1}(\beta)^{\dagger}\bar{Y}_{\nu}^{\dagger}\alpha_{\mathcal{R}}^{\dagger}\tilde{S}^{-1}(\bar{Z}_{\nu}^{\dagger}), \quad (7.45)$$

where

$$\begin{aligned} \bar{X}_{\nu}^{\dagger} \otimes \bar{Y}_{\nu}^{\dagger} \otimes \bar{Z}_{\nu}^{\dagger} &= (\Phi^{\dagger})^{-1} \\ &= \Phi_{\Omega} = (\Omega \otimes 1) \cdot (\Delta \otimes 1)\Omega \cdot \Phi \cdot (1 \otimes \Delta)\Omega^{-1} \cdot (1 \otimes \Omega^{-1}) \\ &= \Omega_i \Omega_j^{(1)} X_{\nu} \bar{\Omega}_k \otimes \Omega^i \Omega_j^{(2)} Y_{\nu} \bar{\Omega}_{(1)}^k \bar{\Omega}_l \otimes \Omega^j Z_{\nu} \bar{\Omega}_{(2)}^k \bar{\Omega}^l \end{aligned}$$

where we have adopted the obvious notation (all repeated indices to be summed over). Substituting into (7.45) then gives

$$\begin{aligned} u^{\dagger} &= \tilde{S}(\Omega_i \Omega_j^{(1)} X_{\nu} \bar{\Omega}_k) \tilde{S}^{-1}(\beta)^{\dagger} \Omega^i \Omega_j^{(2)} Y_{\nu} \bar{\Omega}_{(1)}^k \bar{\Omega}_l \alpha_{\mathcal{R}}^{\dagger} \tilde{S}^{-1}(\Omega^j Z_{\nu} \bar{\Omega}_{(2)}^k \bar{\Omega}^l) \\ &\stackrel{(5.24)(i)}{=} \tilde{S}(\Omega_i \Omega_j^{(1)} X_{\nu} \bar{\Omega}_k) w^{-1} \alpha_{\Omega} \Omega^i \Omega_j^{(2)} Y_{\nu} \bar{\Omega}_{(1)}^k \bar{\Omega}_l \alpha_{\mathcal{R}}^{\dagger} \tilde{S}^{-1}(\bar{\Omega}^l) \tilde{S}^{-1}(\Omega^j Z_{\nu} \bar{\Omega}_{(2)}^k) \\ &\stackrel{(5.24)(ii)}{=} w^{-1} S(\Omega_i \Omega_j^{(1)} X_{\nu} \bar{\Omega}_k) \alpha_{\Omega} \Omega^i \Omega_j^{(2)} Y_{\nu} \bar{\Omega}_{(1)}^k \bar{\Omega}_l \alpha_{\mathcal{R}}^{\dagger} \tilde{S}^{-1}(\bar{\Omega}^l) \tilde{S}^{-1}(\Omega^j Z_{\nu} \bar{\Omega}_{(2)}^k) \\ &= w^{-1} S(\Omega_j^{(1)} X_{\nu} \bar{\Omega}_k) S(\Omega_i) \alpha_{\Omega} \Omega^i \Omega_j^{(2)} Y_{\nu} \bar{\Omega}_{(1)}^k \bar{\Omega}_l \alpha_{\mathcal{R}}^{\dagger} \tilde{S}^{-1}(\bar{\Omega}^l) \tilde{S}^{-1}(\Omega^j Z_{\nu} \bar{\Omega}_{(2)}^k). \end{aligned}$$

Now,

$$S(\Omega_i) \alpha_{\Omega} \Omega^i = (\alpha_{\Omega})_{\Omega^{-1}} = \alpha_{\Omega^{-1}\Omega} = \alpha$$

and from lemma 5(ii)

$$\begin{aligned} \bar{\Omega}_l \alpha_{\mathcal{R}}^{\dagger} \tilde{S}^{-1}(\bar{\Omega}^l) &= S^{-1}(\beta_{\bar{\mathcal{R}}}) S^{-1}(w) \stackrel{(7.44)}{=} S^{-1}[S^{-1}(\beta) \bar{u}^{-1}] \cdot S^{-1}(w) \\ &= S^{-1}[\bar{u}^{-1} S(\beta)] S^{-1}(w) = \beta S^{-1}(\bar{u}^{-1}) S^{-1}(w). \end{aligned}$$

At this point it is worth noting that,

$$S^{-1}(\bar{u}^{-1}) S^{-1}(a) S^{-1}(\bar{u}) = S^{-1}[\bar{u} a \bar{u}^{-1}] \stackrel{(7.44)}{=} S^{-1}[S^2(a)] = S(a). \quad (7.46)$$

Substituting into the above gives

$$\begin{aligned}
u^\dagger &= w^{-1} S(\Omega_j^{(1)} X_\nu \bar{\Omega}_k) \alpha \Omega_j^{(2)} Y_\nu \bar{\Omega}_{(1)}^k \beta S^{-1}(\bar{u}^{-1}) S^{-1}(w) \tilde{S}^{-1}(\Omega^j Z_\nu \bar{\Omega}_{(2)}^k) \\
&\stackrel{(5.26)(iv)}{=} w^{-1} S(\Omega_j^{(1)} X_\nu \bar{\Omega}_k) \alpha \Omega_j^{(2)} Y_\nu \bar{\Omega}_{(1)}^k \beta S^{-1}(\bar{u}^{-1}) S^{-1}(\Omega^j Z_\nu \bar{\Omega}_{(2)}^k) S^{-1}(w) \\
&\stackrel{(7.46)}{=} w^{-1} S(\Omega_j^{(1)} X_\nu \bar{\Omega}_k) \alpha \Omega_j^{(2)} Y_\nu \bar{\Omega}_{(1)}^k \beta S(\Omega^j Z_\nu \bar{\Omega}_{(2)}^k) S(w) S^{-1}(\bar{u}^{-1}) \\
&= w^{-1} S(X_\nu \bar{\Omega}_k) S(\Omega_j^{(1)}) \alpha \Omega_j^{(2)} Y_\nu \bar{\Omega}_{(1)}^k \beta S(\bar{\Omega}_{(2)}^k) S(\Omega^j Z_\nu) S(w) S^{-1}(\bar{u}^{-1}) \\
&\stackrel{(2.6)}{=} w^{-1} S(X_\nu) \alpha Y_\nu \beta S(Z_\nu) S(w) S^{-1}(\bar{u}^{-1}) \\
&\stackrel{(2.6)}{=} w^{-1} S(w) S^{-1}(\bar{u}^{-1}).
\end{aligned}$$

Hence we have proved

Proposition 10.

$$u^\dagger = w^{-1} S(w) S^{-1}(\bar{u}^{-1}).$$

With regard to proposition 10 it is worth noting the following result concerning the antipode \tilde{S} of equation (4.21):

Lemma 6. $v = w^{-1} S(w) u$ determines a u -operator for the quasi-triangular QHA structure of proposition 3', i.e.

$$\tilde{S}(a) = v \tilde{S}^{-1}(a) v^{-1}, \quad \forall a \in H.$$

Proof. From equation (5.24)(ii) we have

$$w \tilde{S}(a) w^{-1} = S(a) = u S^{-1}(a) u^{-1} \stackrel{(5.26)(iv)}{=} u S^{-1}(w) \tilde{S}^{-1}(a) S^{-1}(w^{-1}) u^{-1}$$

which implies

$$\tilde{S}(a) = w^{-1} u S^{-1}(w) \tilde{S}^{-1}(a) S^{-1}(w^{-1}) u^{-1} w, \quad \forall a \in H$$

so that

$$v = w^{-1} u S^{-1}(w) = w^{-1} S(w) u$$

is a u -operator for \tilde{S} . □

Corollary. u^\dagger is also a u -operator for \tilde{S} .

Proof. This follows from lemma 6 and proposition 10 by noting from equation (7.46) above, that $S^{-1}(\bar{u}^{-1})$ is a u -operator for H (with respect to S). □

Following the definition (4.16-4.18) of a $*$ -QHA, it is natural to define a quasi-triangular $*$ -QHA ($*$ -QTQHA) as one for which the complete QHA structure of proposition 3' is obtainable [modulo (S, α, β)] by twisting with Ω . However, since $(\mathcal{R}^T)^{-1}$ is also an R -matrix this leads to two natural classes of $*$ -QTQHA:

Definition 5. A $*$ -QHA with $*$ -canonical element Ω is called a $*$ -QTQHA of Type I (resp. Type II) if it is a quasi-triangular QHA with R -matrix satisfying

$$(\mathcal{R}^\dagger)^{-1} = \Omega^T \mathcal{R} \Omega^{-1}, \quad [\text{resp. } \Omega^T (\mathcal{R}^T)^{-1} \Omega^{-1}]. \quad (7.47)$$

In the Hopf algebra setting Majid [20] has pointed out that two natural Hopf-* structures arise in the quasi-triangular case, called the antireal case, $\mathcal{R}^\dagger = \mathcal{R}^{-1}$, and the real case $\mathcal{R}^\dagger = \mathcal{R}^T$. For *-QTQHA we see that upon setting $\Omega = 1 \otimes 1$ the type I case reduces to Majid's antireal case and the type II to his real case. In the triangular case, corresponding to $\mathcal{R}^T \mathcal{R} = 1 \otimes 1$, or $\mathcal{R}^T = \mathcal{R}^{-1}$, the type I and type II cases coincide.

Our main result here is,

Theorem 9 (Twist Invariance). *Let $F \in H \otimes H$ be an arbitrary twist on a *-QTQHA of type I (resp. type II). Then H is also a *-QTQHA of type I (resp. type II) under the twisted structure induced by F .*

Proof. Following Theorem 1, it suffices to prove that under the twisted structure induced by F equation (7.47) holds: recall that H is also a quasi-triangular QHA under this structure with R -matrix $\mathcal{R}_F = F^T \mathcal{R} F^{-1}$. For the type I case we have

$$\begin{aligned} (\mathcal{R}_F^\dagger)^{-1} &= (F^T \mathcal{R} F^{-1})^{\dagger^{-1}} \\ &= (F^T)^{\dagger^{-1}} (\mathcal{R}^\dagger)^{-1} F^\dagger \\ &= (F^T)^{\dagger^{-1}} \Omega^T \mathcal{R} \Omega^{-1} F^\dagger \\ &= (F^T)^{\dagger^{-1}} \Omega (F^T)^{-1} \mathcal{R}_F F \Omega^{-1} F^\dagger \\ &= \Omega_F^T \mathcal{R}_F \Omega_F^{-1} \end{aligned}$$

where $\Omega_F = (F^\dagger)^{-1} \Omega F^{-1}$ is the *-canonical element for the twisted structure, which shows that H is also a type I *-QTQHA under this structure. The proof is similar for the type II case. \square

Thus in the type I case the R -matrix $\bar{\mathcal{R}}$ of equation (7.43) is given by

$$\bar{\mathcal{R}} = (\Omega^T)^{-1} (\mathcal{R}^\dagger)^{-1} \Omega = (\Omega^T)^{-1} (\Omega^T R \Omega^{-1}) \Omega = R$$

while in the type II case

$$\bar{\mathcal{R}} = (\Omega^T)^{-1} (\mathcal{R}^\dagger)^{-1} \Omega = (\Omega^T)^{-1} (\Omega^T (\mathcal{R}^T)^{-1} \Omega^{-1}) \Omega = (\mathcal{R}^T)^{-1}.$$

Thus for the u -operator \bar{u} of equation (7.44) we have

$$\bar{u} = \begin{cases} u & (\text{type I case}) \\ \tilde{u} & (\text{type II case}) \end{cases}$$

where

$$\tilde{u} = S(Y_\nu \beta S(Z_\nu)) \alpha_{\bar{\mathcal{R}}} X_\nu$$

which was shown in [12] to be given by $\tilde{u} = S(u^{-1})$. In view of proposition 10 we thus arrive at

Proposition 10' *Let H be a *-QTQHA. Then the u -operator of equation (7.42) must satisfy*

$$u^\dagger = w^{-1} S(w) \cdot \begin{cases} S(u^{-1}) & (\text{type I case}) \\ u & (\text{type II case}). \end{cases}$$

For the type I case above, we used the well known result $S(u^{-1}) = S^{-1}(u^{-1})$, as is easily verified.

It is easily verified that

$$z_u \equiv u S(u) = S(u) u \quad (7.48)$$

is a central element, as shown in [13]. In terms of z_u the result of proposition (10') is expressible as

$$u^\dagger = w^{-1}S(w) \cdot \begin{cases} z_u^{-1}u & (\text{type I case}) \\ z_u S(u^{-1}) & (\text{type II case}). \end{cases}$$

The main difference between the type I and type II $*$ -cases lies in the nature of the central element z_u : it is always unitary in the type I case while in the type II case it is self-adjoint. Explicitly

Lemma 7. *Let H be a $*$ -QTQHA and z_u the central element of equation (7.48). Then*

$$z_u^\dagger = \begin{cases} z_u^{-1} & (\text{type I case}) \\ z_u & (\text{type II case}). \end{cases}$$

Proof. First observe from equation (5.26)(iv) that

$$\tilde{S}^{-1}(a) = S^{-1}(w^{-1})S^{-1}(a)S^{-1}(w), \quad \forall a \in H. \quad (*)$$

Now, with $z_u = uS(u)$ we have

$$z_u^\dagger = \tilde{S}^{-1}(u^\dagger)u^\dagger \stackrel{(*)}{=} S^{-1}(w^{-1})S^{-1}(u^\dagger)S^{-1}(w)u^\dagger.$$

Thus in the type I case

$$\begin{aligned} z_u^\dagger &= S^{-1}(w^{-1})S^{-1}[w^{-1}S(w)S(u^{-1})] \cdot S^{-1}(w)w^{-1}S(w)S(u^{-1}) \\ &= S^{-1}(w^{-1})[u^{-1}wS^{-1}(w^{-1})] \cdot S^{-1}(w)w^{-1}S(w)S(u^{-1}) \\ &= S^{-1}(w^{-1})u^{-1}S(w)S(u^{-1}) \\ &= S^{-1}(w^{-1})S^{-1}(w)u^{-1}S(u^{-1}) \\ &= u^{-1}S(u^{-1}) = z_u^{-1}, \end{aligned}$$

while in the type II case

$$\begin{aligned} z_u^\dagger &= S^{-1}(w^{-1})S^{-1}[w^{-1}S(w)u] \cdot S^{-1}(w)w^{-1}S(w)u \\ &= S^{-1}(w^{-1})[S^{-1}(u)wS^{-1}(w^{-1})] \cdot S^{-1}(w)w^{-1}S(w)u \\ &= S^{-1}(w^{-1})S^{-1}(u)S(w)u \\ &= S^{-1}(w^{-1})S^{-1}(w)S^{-1}(u)u \\ &= S^{-1}(u)u = uS(u) = z_u. \end{aligned}$$

□

Lemma 7 holds quite generally, regardless of whether or not Ω is self-adjoint or S is $*$ -compatible. It is a universal property completely independent of Ω .

Theorem 9 shows that the category of type I or type II $*$ -QTQHAs is invariant under twisting. Now H is also a quasi-triangular QHA with R -matrix \mathcal{R}^T under the opposite structure of proposition 1 and is obtainable by twisting with \mathcal{R} . Moreover, proposition 4 shows that H is also a $*$ -QHA under this opposite structure with $*$ -canonical element $\Omega^T = T \cdot \Omega$. It is therefore not surprising that we have the following extension

Proposition 4' *A type I (resp. type II) $*$ -QTQHA is also a type I (resp. type II) $*$ -QTQHA under the opposite structure of proposition 4 with R -matrix $\mathcal{R}^T = T \cdot \mathcal{R}$.*

Proof. In view of the above and Proposition 4 it remains to check equation (7.47) for this opposite structure. To this end we have in the type I case

$$(\mathcal{R}^T)^\dagger{}^{-1} \stackrel{(7.47)}{=} \Omega \mathcal{R}^T (\Omega^T)^{-1} = (\Omega^T)^T \mathcal{R}^T (\Omega^T)^{-1}$$

and similarly for the type II case

$$(\mathcal{R}^T)^\dagger{}^{-1} \stackrel{(7.47)}{=} \Omega \mathcal{R}^{-1} (\Omega^T)^{-1} = (\Omega^T)^T (\mathcal{R}^T)^{T-1} (\Omega^T)^{-1}$$

which proves equation (7.47) for the opposite structure as required. \square

It is important to note that the definition of \ast -QTQHA depends explicitly on the \ast -canonical element Ω which is interconnected with the R -matrix through equation (7.47). Indeed, if $\Omega_1 = \Omega C$ is another \ast -canonical element with C a compatible twist [see Theorem 2] then H will not generally be a \ast -QTQHA with respect to Ω_1 as is easily seen. However, following Theorem 9, H will be a \ast -QTQHA with twisted canonical element $\Omega_C = (C^\dagger)^{-1} \Omega C^{-1}$ and R -matrix $\mathcal{R}_C = C^T \mathcal{R} C^{-1}$.

As noted above, the definition of a \ast -QTQHA depends on the \ast -canonical element Ω (as well as \mathcal{R}). We in fact have the following extension of Theorem 2:

Theorem 2' *Let H be a \ast -QTQHA with \ast -canonical element Ω and R -matrix \mathcal{R} . Then H is also a \ast -QTQHA with the same R -matrix \mathcal{R} but with \ast -canonical element Γ if and only if there exists a compatible twist $C \in H \otimes H$ such that $\Gamma = \Omega C$ and*

$$C^T \mathcal{R} C^{-1} = \mathcal{R}. \quad (7.49)$$

Proof. First from Theorem 2, in order for Γ to be a \ast -canonical element there must exist a compatible twist $C \in H \otimes H$ such that $\Gamma = \Omega C$. Now suppose H is a \ast -QTQHA of type I, so that

$$(\mathcal{R}^\dagger)^{-1} = \Omega^T \mathcal{R} \Omega^{-1}.$$

Then in order for H to be a \ast -QTQHA of type I with respect to Γ it is necessary and sufficient that

$$\begin{aligned} (\mathcal{R}^\dagger)^{-1} = \Gamma^T \mathcal{R} \Gamma^{-1} &\Leftrightarrow \Gamma^T \mathcal{R} \Gamma^{-1} = \Omega^T \mathcal{R} \Omega^{-1} \\ &\Leftrightarrow \Omega^T C^T \mathcal{R} C^{-1} \Omega^{-1} = \Omega^T \mathcal{R} \Omega^{-1} \\ &\Leftrightarrow C^T \mathcal{R} C^{-1} = \mathcal{R} \end{aligned} \quad (7.50)$$

and similarly for the type II case. This proves the result. \square

We thus have

Definition 6. *We call a compatible twist C on a quasi-triangular QHA a quasi-triangular compatible twist if it satisfies equation (7.49).*

Twisting a quasi-triangular QHA with such a twist will leave the entire structure unchanged (modulo α, β). Quasi-triangular compatible twists on a quasi-triangular QHA H form a subgroup of the group of compatible twists on H . Theorem 2' shows that for a given R -matrix there is a 1-1 correspondence between \ast -canonical elements for a \ast -QTQHA H and quasi-triangular compatible twists on H .

It is worth noting that

Lemma 8. *Let (H, Ω, \mathcal{R}) be a \ast -QTQHA. Then H is also a \ast -QTQHA with \ast -canonical element Ω^\dagger .*

Proof. Taking the conjugate inverse of equation (7.47) gives

$$\mathcal{R} = (\Omega^T)^{-1\dagger}(\mathcal{R}^\dagger)^{-1}\Omega^\dagger \quad [\text{resp.}(\Omega^T)^{-1\dagger}(\mathcal{R}^T)^\dagger\Omega^\dagger]$$

or equivalently

$$(\mathcal{R}^\dagger)^{-1} = (\Omega^\dagger)^T\mathcal{R}(\Omega^\dagger)^{-1} \quad [\text{resp.}(\Omega^\dagger)^T(\mathcal{R}^T)^{-1}(\Omega^\dagger)^{-1}]$$

which, together with proposition 2, is sufficient to prove the result. \square

Corollary. $\Omega^{-1}\Omega^\dagger$ must determine a quasi-triangular compatible twist on H .

This last result puts a strong restriction on Ω in order for it to give rise to a *-canonical element for a *-QTQHA.

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